Chapter 1

Classical Mechanics

1.1 Lagrangians

The purpose of quantum mechanics is to calculate the interaction of light, electrons and other particles. Physicists do this by a succession of theories: classical mechanics, Schrödniger quantum mechanics, quantum field theory. It is hoped that string theory will be added to the list. Each theory actually gives good, but not perfect, predictions of experiments over a certain energy range. The theories do not give good results if stretched beyond their energy range.

Their are two formulations of classical mechanics: Lagrangian and Hamiltonian which are useful in quantum mechanics. The Lagrangian formulation is useful for the path integral point of view and the Hamiltonian formulation is useful for the Hilbert space point of view. We begin with the Lagrangian formulation. Suppose we want to describe l particles moving in $\mathbf{R}^n$. The position of the particles can be described as a point in $(\mathbf{R}^n)^l = \mathbf{R}^{nl} = C$. Let T be the tangent bundle of C. A point of T represents the position of a particles and their velocities. Let $\mathcal{L}$ be a real valued function on T. Let P and Q be two points in C and let $\gamma : [0,1] \to C$ be a path from P to Q. There is a natural lift of γ to T which we can write as $t : [0,1] \to (\gamma(t), \gamma'(t))$.

Definition 1.1.0.1. The action of γ is

$$S[\gamma] = \int_0^1 \mathcal{L}(\gamma(t), \gamma'(t))dt \tag{1.1}$$

Definition 1.1.0.2. γ satisfies the equations of motion of classical mechanics if

$$\frac{d}{d\epsilon}S[\gamma+\epsilon\gamma_1]=0 \tag{1.2}$$

whenever $\gamma_1(0)=\gamma_1(1)=0$.

We can write

$$\mathcal{L}=\mathcal{L}(q_1,q_2\ldots,q_{kl};v_1,v_2\ldots) \tag{1.3}$$

These equations of motion can be explicitly written out as the Euler-Lagrange equations of motion.

$$\frac{\partial\mathcal{L}}{\partial q_i}(\gamma(t),\gamma'(t))=\frac{d}{dt}(\frac{\partial\mathcal{L}}{\partial v_i})(\gamma(t),\gamma'(t)) \tag{1.4}$$

1.2 Hamiltonians

Suppose that γ obeys the Euler-Lagrange equations of motion for some Lagrangian $\mathcal{L}$. It turns out that there is always a function

$$H=H(q_1,q_2\ldots;v_1,v_2,\ldots) \tag{1.5}$$

which is conserved, i.e.

$$\frac{d}{dt}(H(\gamma(t),\gamma'(t))=0 \tag{1.6}$$

Specifically,

1.3 Classical fields

The machinery of Lagranigians can generalized to infinite dimensional systems. Here we will consider the simplest case, namely we replace Euclidean space by the space of all functions ϕ on the Euclidean space. Here 'all' has to be taken with a grain of salt. For our purposes, we can often consider $\mathcal{C}^\infty$ functions with compact support. Suppose ϕ is a function on Minkowski space. The Lagrangian of the theory is generally just a local expression in ϕ and the partials of ϕ, usually just the first partials. A non-trivial example is

$$\mathcal{L}(\phi)(x) = \sum \left(\frac{\partial \phi}{\partial x_i} \right)^2 + m\phi(x)^2 + \epsilon \phi(x)^4 \tag{1.7}$$

Associated to a Lagrangian $\mathcal{L}$ is the action S. Given a $\mathcal{C}^\infty$ function ψ which dies rapidly at infinity, we can form

$$S(\phi) = \int_{\mathbf{R}^n} \mathcal{L}(\phi) \tag{1.8}$$

To a Lagrangian $\mathcal{L}$, we can associate a differential operator called the Euler-Lagrange operator of $\mathcal{L}$ in the following way: Let ψ be a $\mathcal{C}^\infty$ function. We can consider the expression:

$$\Phi_{\mathcal{L}}(\psi, \phi) = \left(\frac{d}{dh} \int \mathcal{L}(\psi + h\phi) - \mathcal{L}(\psi)\, dx \right)_{h=0} \tag{1.9}$$

for any ϕ with compact support. Then there is a differential operator $E_{\mathcal{L}}$ so that $E_{\mathcal{L}}(\psi) = 0$ if and only if $\Phi_{\mathcal{L}}(\psi, \phi) = 0$ for all ϕ with compact support. E is called the Euler-Lagrange equation associated to $\mathcal{L}$. of course, this property does not define the Euler-Lagrange equation uniquely, but there is a recipe for computing the Euler-Lagrange equation. Namely, we can calculate

$$\mathcal{L}(\psi + h\phi) - \mathcal{L}(\psi) \tag{1.10}$$

to first order in h. The result will only contain terms linear in ϕ and the derivatives of ϕ. Using integration by parts, any occurrences of the partials of ϕ can be replaced by ϕ. Then end result will be $\int \phi(x) E_{\mathcal{L}}(x)$ for some differential operator $)E_{\mathcal{L}}$.

For instance, suppose

$$\mathcal{L}(\phi)(x) = \sum \left(\frac{\partial \phi}{\partial x_i} \right)^2 \tag{1.11}$$

Then to first order in h, we have

$$\mathcal{L}(\psi + h\phi) - \mathcal{L}(\psi) = 2h \sum \left(\frac{\partial \psi}{\partial x_i} \right) \left(\frac{\partial \phi}{\partial x_i} \right) + \dots \tag{1.12}$$

But

$$\int \left(\frac{\partial\psi}{\partial x_i}\right)\left(\frac{\partial\phi}{\partial x_i}\right) = \int -\phi \left(\frac{\partial^2\psi}{\partial x_i^2}\right) \tag{1.13}$$

So

$$E_{\mathcal{L}}(\psi) = -\sum \left(\frac{\partial^2\psi}{\partial x_i^2}\right) \tag{1.14}$$

We will assume that the Lagrangian only depends on ϕ and the first partials of ϕ.

$$\mathcal{L}(\phi, \partial_1(\phi), \partial_2(\phi) \dots), \tag{1.15}$$

where $\partial_k(\phi)$ indicates the partial of ϕ with respect to the k^{th} variable. Then we denote

$$\frac{\partial L}{\partial(\partial_k(\phi))} \tag{1.16}$$

the result of taking $\partial_{k+1}\mathcal{L}$ as an ordinary function and then plugging in $(\phi, , \partial_1(\phi), \partial_2(\phi) \dots)$, e.g.

$$\frac{\partial L}{\partial(\partial_k(\phi))} = (\partial_{k+1}\mathcal{L})(\phi, \partial_1(\phi), \partial_2(\phi) \dots) \tag{1.17}$$

Similarly,

$$\frac{\partial L}{\partial\phi} = (\partial_1\mathcal{L})(\phi, \partial_1(\phi), \partial_2(\phi) \dots) \tag{1.18}$$

With this notation, the Euler-Lagrange equations become

$$\sum_k \partial_k \left(\frac{\partial\mathcal{L}}{\partial(\partial_k(\phi))}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \tag{1.19}$$

The usual Einstein convention is to sum over any pair of repeated indices, so this becomes

$$\partial_k \left(\frac{\partial\mathcal{L}}{\partial(\partial_k(\phi))}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \tag{1.20}$$

Note that passing from the Lagrangian to the Euler-Lagrange equations is a completely algebraic operation. Namely, write out the expression

$$\mathcal{L}(\psi + \epsilon\phi, \partial_1(\psi + \epsilon\phi), \partial_2(\psi + \epsilon\phi), \dots) \tag{1.21}$$

to first order in ϵ. Then eliminate any expressions in $F_k(\psi)\partial_k(\phi)$ using the expression

$$-\partial_k(F_k(\psi))\phi. \tag{1.22}$$

The Euler-Lagrange equations are then the coefficient to ϕ in the resulting expression.

1.4 Conserved quantities

Let $\mathcal{V}$ denote the space of $\mathcal{C}^\infty$ functions on $\mathbf{R}^n$. Let $\Phi_t : \mathcal{V} \to \mathcal{V}$ for $t \in \mathbf{R}$ be a one dimensional family of self maps of $\mathcal{V}$ with Φ_0 being the identity. We assume that Φ is a homomorphism,

$$\Phi_{t+t'} = \Phi_t \Phi_{t'} \tag{1.23}$$

We have in mind not some abstract setting but a reasonable class of Φ_t which represents some symmetry of the action. For instance, given a vector $v \in \mathbf{R}^n$, let

$$\Phi_t(\phi)(x) = \Phi(x + tv) \tag{1.24}$$

Given an $n \times n$ matrix A, another example would be

$$\Phi_t(\phi)(x) = \phi(\exp(tA)x) \tag{1.25}$$

First, we assume the Φ_t are symmetries of the action S (1.8), i.e.

$$S(\Phi_t(\phi)) = S(\phi) \tag{1.26}$$

Now Noether's theorem states that if the Φ_t are all symmetries of the action and ϕ satisfies the Euler-Lagrange equations (1.20), then we can construct a one form ω_ϕ on $\mathbf{R}^n$ with the divergence of ω_ϕ being zero. The components of ω_ϕ will be computed locally from ϕ in terms of ϕ and the partials of ϕ. Now associated to a one form, there is the dual $n-1$ form ω'_ϕ and the divergence condition corresponds to $d\omega'_\phi = 0$, i.e. ω'_ϕ is closed.

We will construct this form below, but let's consider why a closed $n-1$ form ω is considered give rise to a conserved quantity. Let's suppose that we interpret x_n as time t. Let's suppose that we are given a ϕ which dies off rapidly at infinity on the hyperplanes $H_a = \{t = a\}$. Then so does ω_ϕ as we shall see in the examples and then

$$V_a = \int_{H_a} \omega'_\phi \tag{1.27}$$

is independent of a by Gauss's theorem, i.e. V_a is conserved.

Here is the construction of ω_ϕ. To first order in t about $t = 0$, we can compute

$$\Phi_t(\phi) = \phi + t\Delta\phi + \dots \tag{1.28}$$

Now up to first order in t, we have

$$\begin{aligned}\mathcal{L}(\Phi_t(\phi)) - \mathcal{L}(\phi) &= t\left(\frac{\partial \mathcal{L}}{\partial \phi}\Delta\phi + \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\partial_\mu(\Delta\phi)\right) + \cdots \\ &= t\left(\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) + \left[-\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu(\phi))}\right) + \frac{\partial \mathcal{L}}{\partial \phi}\right]\right) \\ &\quad)\end{aligned}$$

So let's define

$$\Psi(\phi) = \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) + \left[-\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu(\phi))}\right) + \frac{\partial \mathcal{L}}{\partial \phi}\right]\Delta\phi, \tag{1.29}$$

which we think of as a differential polynomial in ϕ. The assumed invariance of the action means that

$$\int \Psi(\phi) = 0 \tag{1.30}$$

for every ϕ for which 1.30 makes sense. The only way this can happen is that we can find differential polynomials $\mathcal{T}^\mu(\phi)$ so that

$$\Psi = \partial_\mu \mathcal{T}^\mu \tag{1.31}$$

as differential polynomials. Let's define

$$j^{\mu}_{\mathcal{L},\Phi}(\phi) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi - \mathcal{T}^{\mu} \tag{1.32}$$

Then

$$\partial_\mu j^{\mu}_{\mathcal{L},\phi} = \left[-\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu(\phi))} \right) + \frac{\partial \mathcal{L}}{\partial \phi} \right] \tag{1.33}$$

for any $\mathcal{C}^{\infty}$ ϕ. So if ϕ satisfies the Euler-Lagrange equations, we get a conservation law:

$$\partial_\mu j^{\mu}_{\mathcal{L},\Phi} = 0 \tag{1.34}$$

Let's work out the example of translation symmetry from 1.24 with $v = (1, 0, 0, \dots)$ and the Lagrangian

$$\mathcal{L}(\phi)(x) = \sum \left(\frac{\partial \phi(x)}{\partial x_i} \right)^2 + m\phi(x)^2 \tag{1.35}$$

Here

$$\Delta(\phi) = \partial_1(\phi) \tag{1.36}$$

so we have to write

$$2\partial_\mu\left(\phi_\mu)\partial_1(\phi)\right) + \left[-2\partial_\mu\partial_\mu\phi + m\phi\right]\partial_1(\phi) \tag{1.37}$$

as a divergence (See equation 1.29)

Chapter 2

Klein-Gordon

2.1 Relativistic conventions

Let

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The physicist like to let the Greek letters run over 0, 1, 2, 3, where 0 represents time t and 1,2,3 represent x, y and z. Then

$$x^\mu = (x_0, x_1, x_2, x_3)$$

and

$$x_\mu = g_{\mu\nu} x^\nu = \sum_\nu g_{\mu\nu} x^\nu.$$

Further,

$$p \cdot x = g_{\mu\nu} x^\nu x^{mu} p^\nu.$$

Also

$$\partial_\mu = \frac{\partial}{\partial x_\mu}.$$