Variable Order Dynamic Models in Anomalous Diffusion Modelling?

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OPEN ISSUES IN ANOMALOUS DIFFUSION MODELING

1. **How to deal with the diffusion process which changes with time evolution, spatial variation or system parameters?**
2. **How to deal with the diffusion process in the inhomogeneous or anisotropic medium?**
3. **How to estimate or model the diffusion process in the oscillating external field?**
4. **Which are the best differential equation models for the description of multi-scale diffusion or multi-field coupling transport system?**
5. **The link in differential equation, statistical methods and the continuous time random walk model (CTRW)?**

In this work, several fractional differential models to address the above problems are suggested. We also attempt to establish the possible link between differential equations, statistical methods and CTRW.

ABSTRACT

Anomalous diffusion phenomena are observed in various physical, chemical and biological situations, which have motivated the development of new mathematical and physical models. In this poster, we present an introduction and discussion of the variable order differential operator in anomalous diffusion modeling. The physical behaviors of the new model, in contrast to normal diffusion, change with the time evolution, spatial variation or other independent functions. In addition, dynamic models with random order, fuzzy order and even dynamical order may be attempted, and the potential applications of each model are highlighted. Based on the consideration of distributed order model, variable order model and variable coefficient model, we proposed a more generalized distributed-variable order model. Finally, some remarks are made on the relationship between differential equations, statistical methods and CTRW.

DIFFERENT VARIABLE ORDER MODELS

The **variable order** time derivative anomalous diffusion model

$$D_t^{\alpha(x,t)}c(x,t) = K\frac{\partial^2 c(x,t)}{\partial x^2},\ 0<\alpha(x,t)<1.$$

In the above expression, from the original source which yields the variable order, the time derivative order $\alpha(x,t)$ can be a function of time, space, concentration or system parameters. This type of model can depict the diffusion process in which the diffusion rate changes with time, space, concentration or system parameters.

The **random order** time derivative anomalous diffusion model

$$D_t^{\alpha_0+\varepsilon_{x,t}}c(x,t) = K\frac{\partial^2 c(x,t)}{\partial x^2},\ p(\varepsilon_{x,t}\left|0<\alpha_0+\varepsilon_{x,t}<1\right.)=1.$$

In the above equation $\varepsilon_{x,t}$ is random variable which changes with time evolution or spatial variation. The change of the time derivative order is caused by the oscillating external field or other perturbation sources.

The **fuzzy order** time derivative anomalous diffusion model

$$D_t^{\alpha(t)}c(x,t) = K\frac{\partial^2 c(x,t)}{\partial x^2},\ \alpha'(t)=\sum_{i=1}^{r}h_i(\alpha(t))g_i\alpha(t),\ \sum_{i=1}^{r}h_i(\alpha(t))=1.$$

This model imply that we just know the initial value of the derivative order and the order interval. The time derivative order is determined by a fuzzy system.

COMPARISON AND DISCUSSION

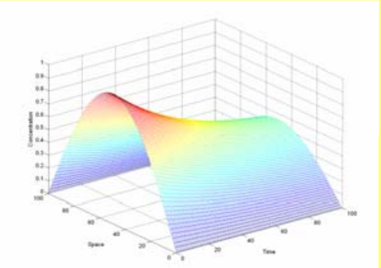

Fig.1 The time dependent variable model

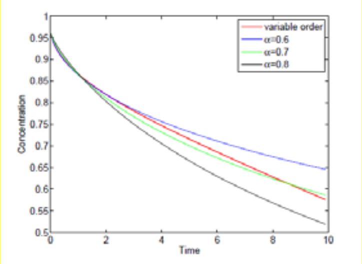

Fig.2 The concentration evolution of time dependent variable model at x=0.6.

$$\begin{cases} D_t^{\alpha(t)}c(x,t)=K\frac{\partial^2 c(x,t)}{\partial x^2}, & 0<x<L,\\ c(x,0)=\sin(\frac{x\pi}{L}), & 0\le x\le L,\\ c(0,t)=c(L,t)=0, & t>0 \end{cases}$$

$$\alpha(t)=\alpha_0+\frac{0.2t}{T_{\max}}$$

α_0=0.6; T_{max} = 10

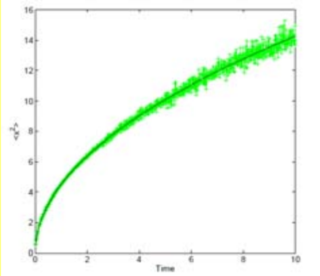

Fig.3 The MSD behavior of random order model

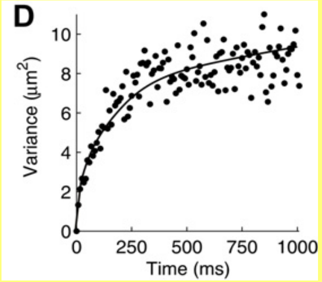

Fig.3 A plot from Ref. [3]

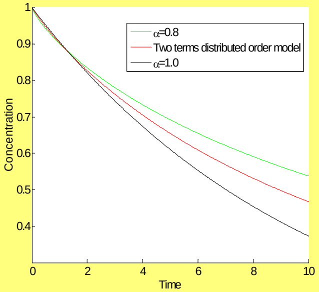

Fig. 4 The behavior of two-term distributed order model

The generalized Distributed-Variable order model

$$\int_0^1 p(\alpha(x,t))D_t^{\alpha(x,t)}c(x,t)d\alpha(x,t)=\frac{\partial}{\partial x}(K(x,t)\frac{\partial c(x,t)}{\partial x}),\ \int_0^1 p(\alpha(x,t))d\alpha(x,t)=1,\ 0<\alpha(x,t)<1.$$

CONCLUSION

- ❖ Variable order model: evolution diffusion process/ corresponding to the CTRW model with variable waiting time PDF;
  Distributed order model: coupling diffusion process/ multi-power law phenomena;
- ❖The variable order model is a more suitable model to characterize the decelerating/accelerating diffusion process.
- ❖The random order model, fuzzy order model and dynamic order model are the special cases of the variable order model.
- ❖ Further research in the area is aimed at providing the CTRW models and statistical methods corresponding to the variable order models.
  The application research of the variable order model is underway, such as bacteria chemotaxi behavior quantification.

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