Laplacian Eigenvalues and Eigenfunctions:
Theory, Computation, Application

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Spectral properties of Lamé operator in Hölder domains

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Cuspoidal domain

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary $\partial\Omega$, with an exception of the origin. We assume that in the neighborhood of the origin the boundary of Ω coincides with α-cusp

$$\Omega \cap \{x_1 < T\} \cap \{|x| < 1\} = \{(x_1, x_2) \in \mathbb{R}^2 : \quad 0 < x_1 < T, \quad |x_2| < x_1^\alpha\}.$$

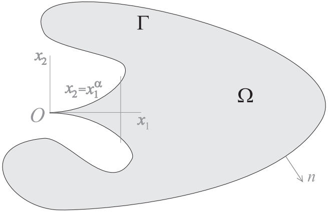


Figure: The domain Ω with an α-cusp.

We consider the case $\alpha > 1$. Let $n(x)$ denote a unit outward normal to $\Gamma = \partial\Omega \backslash \{0\}$.

Elastic problem in cuspoidal domain

Let us consider the boundary value problem

$$\mathcal{L}u = f \quad \text{in} \quad \Omega, \qquad \mathcal{N}u = g \quad \text{on} \quad \Gamma, \tag{1}$$

for the Lamé operator of linear elasticity theory

$$\mathcal{L} = \mu\Delta u + (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div},$$

and with no-traction boundary conditions of the form

$$\mathcal{N}u = \mu\partial_n u + \lambda(\mathrm{div}\, u)n + \mu(\mathrm{grad}\, u)^T n,$$

with ∂_n being a normal derivative operation applied to each component of the vector. Here $u = u(x) = (u_1(x), u_2(x))$ is a displacement vector, $\lambda \geq 0$ and $\mu > 0$ are Lamé constants.

We are interested in the behaviour of solutions arising from the presence of α-cusp, $\alpha \geq 2$. This question is addressed by studying the spectral properties of the problem

$$\mathcal{L}u = \omega^2 u \quad \text{in} \quad \Omega, \qquad \mathcal{N}u = 0 \quad \text{on} \quad \Gamma. \tag{2}$$

Elastic energy

Considering variational formulation, we have

$$(\mathcal{L}u, v)_\Omega = E(u, v) - (\mathcal{N}u, v)_{\partial\Omega}$$

with a scalar product $(u, v)_\Omega = \int_\Omega u \cdot \overline{v}\, dx$ and "elastic energy" E given by $E(u, v) = \int_\Omega \sigma_{ij}\overline{\epsilon}_{ij} dx$, which connects the strain tensor

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

and the stress tensor

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\delta_{ij}\epsilon_{kk}.$$

Note that

$$\begin{aligned} E(u, u) &= \int_\Omega 2\mu|\partial_1 u_1|^2 + 2\mu|\partial_2 u_2|^2 + \mu|\partial_1 u_2 + \partial_2 u_1|^2 + \lambda|\partial_1 u_1 + \partial_2 u_2|^2 dx \\ &\sim \int_\Omega |\partial_1 u_1|^2 + |\partial_2 u_2|^2 + |\partial_1 u_2 + \partial_2 u_1|^2 dx. \end{aligned}$$

Remarks on the Neumann Laplacian

- If Ω is bounded and $\partial\Omega$ is smooth then the Neumann Laplacian problem $-\Delta u = \omega^2 u$ in Ω, $\quad \partial_n u = 0$ on $\partial\Omega$ has a standard discrete spectrum
- The discreteness is a consequence of the compact embedding $H^1(\Omega) \subset L_2(\Omega)$.
- There are examples of bounded domains providing not standard spectrum (e.g. Nikodým, V. Maz'ya, B. Simon, B. Davies)
- Domains with cusps are "nice" domains for Neumann Laplacian, however the presence of the cusp may destroy the spectral asymptotics (see Y. Netrusov, Y. Safarov 2005)
- In unbounded domains an essential spectrum appears. For instance, if $\Omega = (0, +\infty) \times (0, \pi)$ then for $0 < \omega^2 < 1 \quad \exists u$ solving $-\Delta u = \omega^2 u$ in Ω, $\partial_n u = 0$ on $\partial\Omega$ such that

$$u = e^{-i\omega x_1} + S e^{i\omega x_1} + \tilde{u}, \quad \tilde{u} \in H^1(\Omega)$$

where S is a scattering matrix. Then a solution of the correspondent non-uniform problem (with a right-hand side) can be fixed only after radiation conditions are applied.

On the Korn inequality

- For the Neumann Laplacian in a "nice" domain, the compactness of embedding $H^1(\Omega) \subset L_2(\Omega)$ is guaranteed by the Korn inequality (K. Friedrichs 1937; G. Duvaut, J.L. Lions 1976)

$$\|u\|_{H^1}^2 \leq c\left(E(u,u) + \|u\|_{L_2}^2\right), \quad u \in V$$

being valid on the energy space

$$V = \{u : \quad E(u,u) + \|u\|_{L_2}^2 \leq \infty\}.$$

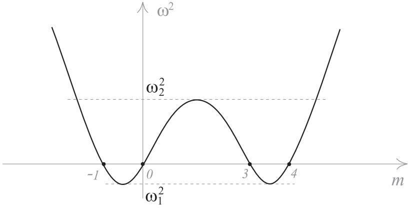
For the Neumann Laplacian $E(u,u) = \int_\Omega |\nabla u|^2 dx$

- For the Lamé operator in Hölder domains the Korn inequality is not valid in general (see Weck, Geymonat & Gilardi 1998). However the embedding $V \subset L_2$ is compact for the cusps with $1 < \alpha < 2$ (N. Weck 1994). Consequently, the spectrum is standard discrete in case $1 < \alpha < 2$.

Results. Critical quadratic cusp: $\alpha = 2$

For $\omega^2 \neq \omega_k^2$ ($k = 1, 2$) there are 4 "even" solutions
$u_j(x) \sim x_1^{-m_j} \binom{0}{1} + \mathcal{O}(x_1^{1-m_j})$ as $x_1 \to 0$ with $m = m_j$ such that

$$(m-4)(m-3)m(m+1)\frac{4\mu(\lambda+\mu)}{3(\lambda+2\mu)} = \omega^2.$$



The lowest value, for which the real solutions exist, is

$$\omega_1^2 = -\frac{21}{4}\frac{\mu(\lambda+\mu)}{\lambda+2\mu}$$

providing two solutions $m = -1/2$ and $m = 7/2$. Note that the point with three real solutions is

$$\omega_2^2 = \frac{75}{4}\frac{\mu(\lambda+\mu)}{\lambda+2\mu}$$

providing $m = 3/2$ and $m = \frac{3 \pm \sqrt{34}}{2}$.

2 "odd" solutions $u_j(x) \sim x_1^{-m_j} \binom{1}{0} + \dots$ as $x_1 \to 0$ with $m_5 = 0$, $m_6 = 1$.
(for $\omega^2 = 0$ the result was known from V. Mazya, A. Soloviev 2001)

Supercritical cusp: $\alpha > 2$

The KERNEL of the operator ($\omega^2 = 0$) consists of
4 "even" solutions

$$
\begin{aligned}
u_1 &\sim x_1 \qquad \varphi_0 + \mathcal{O}(x_1^{\alpha}),\\
u_2 &\sim \qquad\quad \varphi_0 + \mathcal{O}(x_1^{\alpha-1}),\\
u_3 &\sim x_1^{3-3\alpha}\varphi_0 + \mathcal{O}(x_1^{2-2\alpha}),\\
u_4 &\sim x_1^{2-3\alpha}\varphi_0 + \mathcal{O}(x_1^{1-2\alpha}), \quad x_1 \to 0
\end{aligned}
$$

with $\varphi_0 = \binom{0}{1}$,

and 2 "odd" solutions

$$
\begin{aligned}
u_5 &\sim \qquad\quad \phi_0 + \mathcal{O}(x_1^{\alpha-1}),\\
u_6 &\sim x_1^{1-\alpha}\phi_0 + \mathcal{O}(\log x_1), \quad x_1 \to 0
\end{aligned}
$$

with $\phi_0 = \binom{1}{0}$.

Continuous spectrum ($\alpha > 2$)

subtle analysis leads to the following representation of 4 "even" solutions

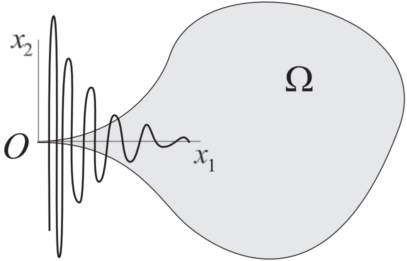
$$u_k(x) \sim \exp\left(p_k \frac{\theta}{\gamma} x_1^{-\gamma}\right) \begin{pmatrix} -p_k x_1^{-\frac{3}{4}\beta} \\ \theta x_1^{-\frac{\beta}{4}} \end{pmatrix} + \dots$$

with

$p_k = (\sqrt[4]{1})_k$: $\quad p_1 = 1,\ p_2 = -1,\ p_3 = i,\ p_4 = -i,$

$\frac{1}{\alpha} + \frac{1}{\beta} = 1$: $\quad 1 < \beta < 2,$

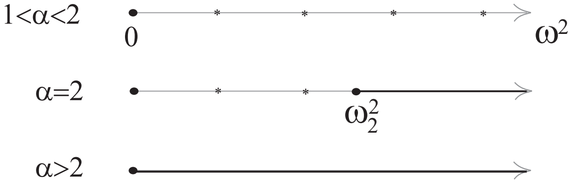
$\gamma = 1 - \frac{\beta}{2}$,
$\theta = \sqrt[4]{J_*}$,
$J_* = \frac{3\omega^2(\lambda+2\mu)}{4\mu(\lambda+\mu)\nu^2}$,
$\nu = (\alpha - 1)^{\beta}$.
Solutions u_5 and u_6 are as above.



Presence of oscillations at all scales !!!
Amplitude and phase increase closer to the origin.

Conclusions

Essential spectrum appears for cusps with $\alpha \geq 2$



For critical ($\alpha = 2$) and supercritical ($\alpha > 2$) cusps the cuspoidal point is playing a role of infinity: essential spectrum, necessity to consider radiation conditions for non-uniform problems.