Hypersurface L^p Estimates for

Approximate Eigenfunctions

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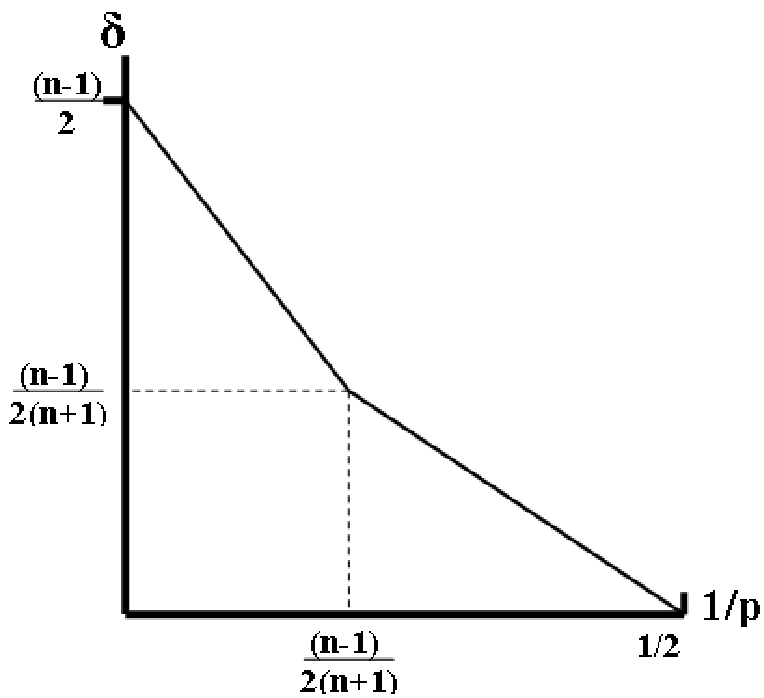
of a Differential Operator

Eigenfunction Estimates

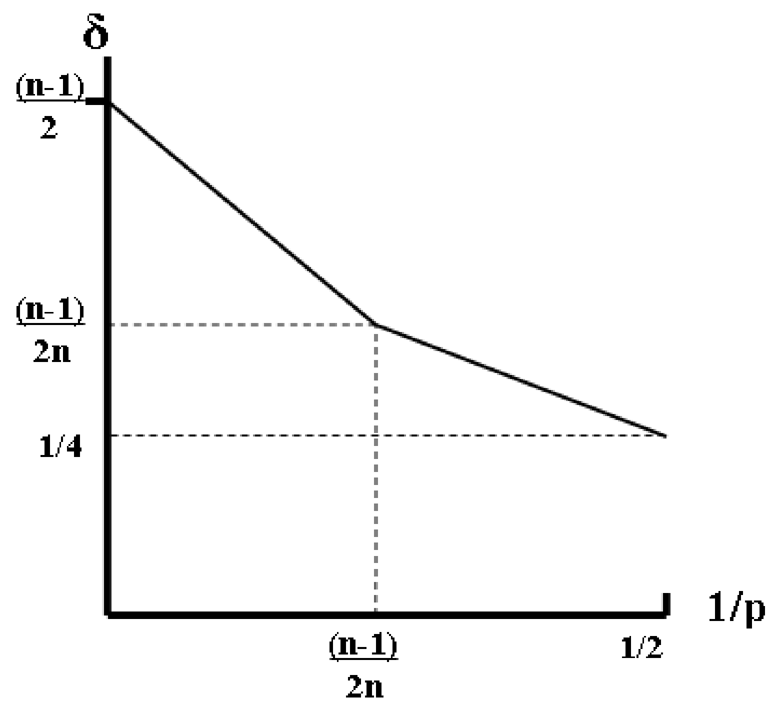
Let u_j an eigenfuction of a differential operator, $Pu_j = \lambda_j u_j$. What is the L^p norm of an eigenfunction restricted to a hypersurface, H, of the manifold, M? We look for a $\delta(p)$ such that

$$Pu_j = \lambda_j^2 u_j \quad \|u_j\|_{L^p(H)} \lesssim \lambda^{\delta(p)} \|u_j\|_{L^2(M)}$$

Sogge's[4] elliptic whole manifold eigenfunction estimates (1988)

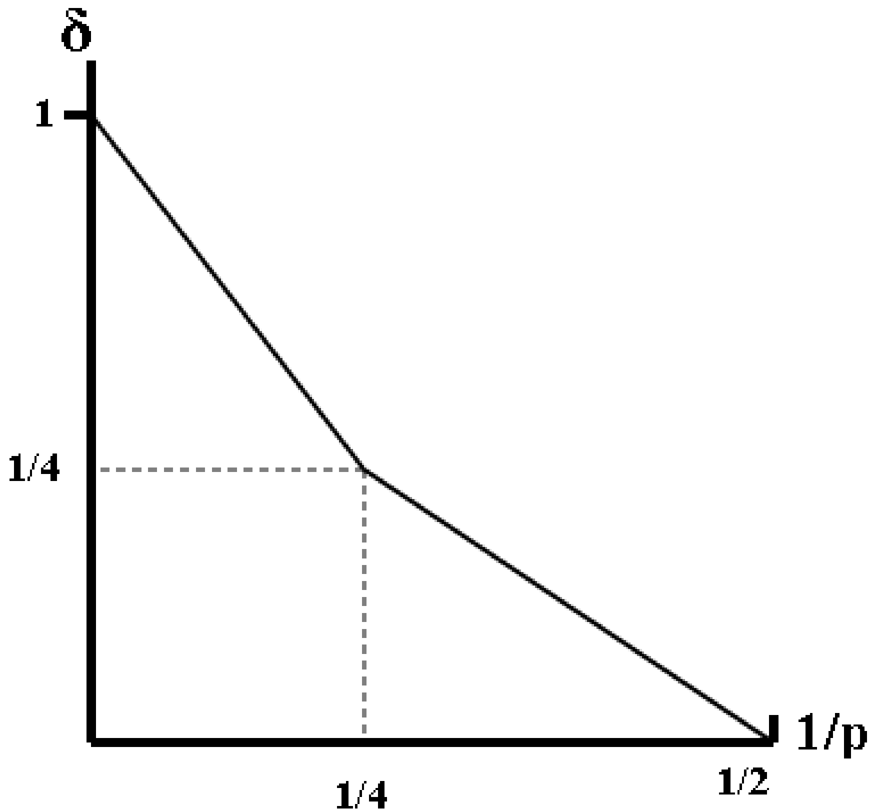


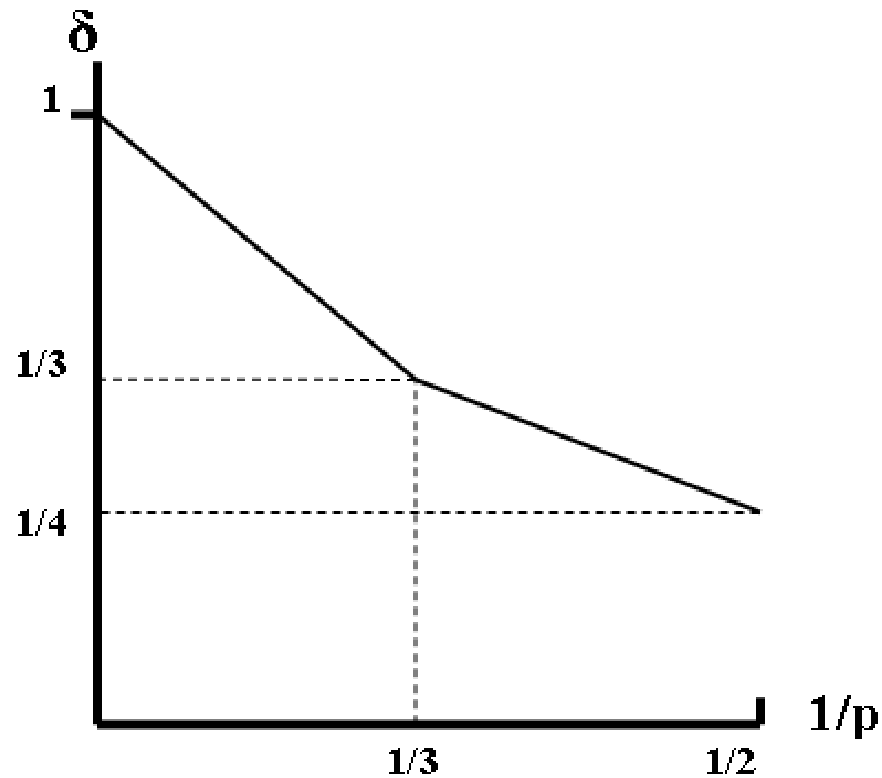
Burq Gérard and Tzetkov's[1] Laplacian hypersurface estimates (2007)



Graphs show $\delta(p)$ plotted against $1/p$ for both cases. Note the similar shape but the kink in the graph is at different values of p.

Example, n=3





These are the estimates stated explicitly when $n = 3$. Here the kink is at $p = 4$ for the whole manifold estimates and $p = 3$ for the hypersurface.

Approach to Generalise Hypersurface Results

- Work in semiclassical regime and turn eigenfunction problem into the statement $Pu = O_{L^2}(h)$ where h is a small parameter. This is what is meant by approximate eigenfunction.
- Use the symbol of P, $p(x, \xi)$ to turn the problem into one about an evolution equation where one of the spacial variables acts as time.
- Use Strichartz estimates to find $L_t^r L_x^p$ estimates then find where $r = p$ (as time in this problem is actually a spatial variable we want to give equal weighting to space and time).
- Interpolate between the L^∞ and L^2 (both found directly) and the L^p norm obtained from the Strichartz estimates to get bounds for intermediate values of p.

Semiclassical Formalism

Sogge's result extended into semiclassical regime by Koch, Tataru and Zworski[3]

Differential Operator → Semiclassical Operator

$$(\Delta + V(x))u = \lambda^2 u$$

$$\lambda^{-2}\Delta u + (\lambda^{-2}V(x) - 1)u = 0$$

Set $\lambda^{-1} = h$ and define the symbol via $hD \rightarrow \xi$

$$p(x, \xi) = \xi^2 + (h^2 V(x) - 1)$$

Theorem Statement

Theorem 1. *Let H be a hypersurface of a compact manifold M. Let $u(h)$ be a family of L^2 normalised functions such that $Pu = O(h)_{L^2}$. Assume further that the symbol of P, $p(x,\xi)$ satisfies non-degeneracy conditions and that $u(h)$ are localised. Then*

$$\|u\|_{L^p(X)} \lesssim h^{-\delta(p)}$$

$$\delta(p) = \begin{cases} \frac{n-1}{2} - \frac{1}{p}, & \frac{2n}{n-1} \leq p \leq \infty \\ \frac{n-1}{4} - \frac{n-2}{2p}, & 2 \leq p \leq \frac{2n}{n-1} \end{cases}$$

- u are 'approximate' eigenfunctions as h is a small parameter
- Localisation is in both space and frequency
- Non-degeneracy assumptions $p(x,\xi) = 0 \Rightarrow \partial_\xi p(x,\xi) \neq 0$ and $\partial^2_{\xi_i\xi_j} p(x,\xi)$ is a non-degenerate matrix We can

  localise around points (x_0,ξ_0) such that $p(x_0,\xi_0) = 0$ as if $p(x_0,\xi_0) \neq 0$, $e^{i\frac{\xi}{h}x}$ is not a good local solution and does not make a large contribution to u if $Pu = O(h)$.

Associated Evolution Equation

Use Implicit Function Theorem to transform symbol into

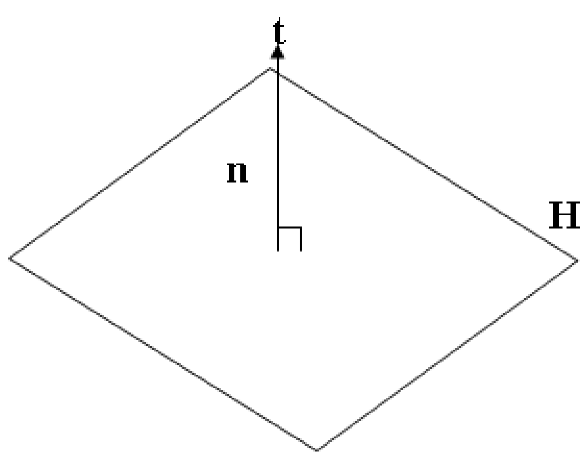
$$p(x,\xi) = e(x,\xi)(\xi_1 - a(x,\xi')) \quad e(x,\xi)\text{approx. invertible}$$

Study evolution equation

$$hD_t - A(t,x,hD_x) \rightarrow U(t) \text{ Evolution Operator}$$

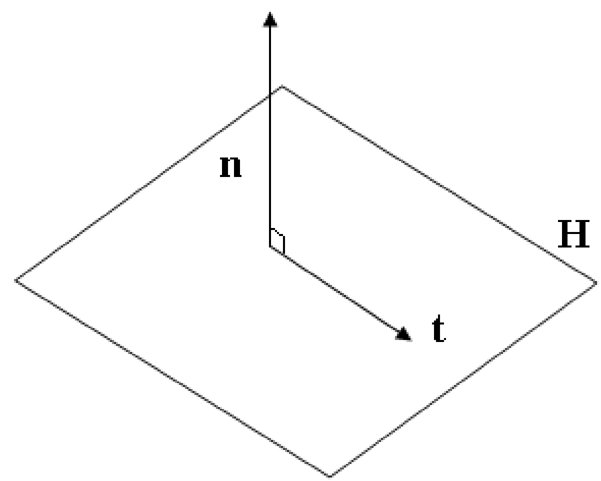
Can show that $\|U(t)u\|_{L^2} \lesssim \|u\|_{L^2}$

$\partial_n p(x,\xi) \neq 0$



$u|_H$ is a timeslice and since $\|U(t)u\|_{L^2} \lesssim \|u\|_{L^2}$ we have $\|u\|_{L^2(H)} \lesssim 1$

$\partial_n p(x,\xi) = 0$



Need to take a $L^p_t L^p_x$ norm as t runs along the hypersurface. Use Strichartz estimates to determine $\|u\|_{L^p(H)}$

Abstract Strichartz Estimates; Keel Tao[2]

Let $U(t) : H \to L^2(X)$ where H is a Hilbert Space and

$$\|U(t)f\|_{L^2(X)} \lesssim \|f\|_H \tag{1}$$

$$\|U(t)U^\star(s)g\|_{L^\infty(X)} \lesssim h^{-\sigma}(h + |t-s|)^{-\sigma} \|g\|_{L^1(X)} \tag{2}$$

then

$$\|U(t)f\|_{L_t^r L_x^p} \lesssim h^{-1/r} \|f\|_H \quad \text{for} \quad \frac{1}{r} + \frac{\sigma}{p} = \frac{\sigma}{2}$$

Turn this problem into one about bilinear forms. Need to prove

$$\left| \iint \langle U(s)^\star F(s), U(t)^\star G(t) \rangle \, dsdt \right| \lesssim \|F\|_{L^{r'}L^{p'}} \|G\|_{L^{r'}L^{p'}}$$

First prove estimate

$$|\langle U(s)^\star F(s), U(t)^\star G(t) \rangle| \lesssim h^{-\gamma}(h + |t-s|)^{-\gamma} \|F(s)\|_{L^{p'}} \|G(t)\|_{L^{p'}}$$

Interpolate between the bilinear forms of(1) and (2) to find γ. Having found γ use Hardy-Littlewood-Sobolev to estimates the t and s integrations. Combine numerologies to get governing equation.

Extended Strichartz Estimates

We are restricting $U(t)$ to a hypersurface so estimates (1) does not hold. Instead we will assume a L^2 bound similar to the decay bound (2).

Let $U(t) : H \to L^2(X)$ where H is a Hilbert Space and

$$\|U(t)U^\star(s)f\|_{L^2(X)} \lesssim h^{-\sigma_2}(h+|t-s|)^{-\sigma_2}\|f\|_{L^2(X)} \tag{3}$$

$$\|U(t)U^\star(s)g\|_{L^\infty(X)} \lesssim h^{-\sigma_\infty}(h+|t-s|)^{-\sigma_\infty}\|g\|_{L^1(X)} \tag{4}$$

then

$$\|U(t)f\|_{L^r_tL^p_x} \lesssim h^{-1/r}\|f\|_H \quad \text{for} \quad \frac{1}{r}+\frac{1}{p}(\sigma_\infty-\sigma_2)=\frac{\sigma_\infty}{2}$$

Again use bilinear forms so we must prove

$$|\langle U(s)^\star F(s), U(t)^\star G(t)\rangle| \lesssim h^{-\gamma}(h+|t-s|)^{-\gamma}\|F(s)\|_{L^{p'}}\|G(t)\|_{L^{p'}}$$

Use the same proceedure as before except this time the L^2 bound (3) has $\gamma=\sigma_2$ rather than $\gamma=0$, the decay bound (4) is unchanged. This changed estimates gives a different governing equation.

Estimates

Write $U(t)$ as a Fourier Integral Operator

$$U(t)f = \frac{1}{(2\pi h)^{n-1}} \iint e^{\frac{i}{h}(\phi(t,x,\eta)-y\cdot\eta)} b(t,x,\eta,h) f(y) dy d\eta$$

Define $W(t)$ by $W(t)f = U(t)f|_H$ Note that as $h \to 0$, $1/h$ becomes large so we can use phase oscillations to get bounds. This is where we use the second non-degeneracy condition that $\partial^2_{\xi_i \xi_j}$ is a non-degenerate matrix.

$$\|W(t)W^\star(s)f\|_{L^2} \lesssim h^{-\frac{1}{2}}(h+|t-s|)^{-\frac{1}{2}} \|f\|_{L^2}$$

$$\|W(t)W^\star(s)f\|_{L^\infty} \lesssim h^{-\frac{n-1}{2}}(h+|t-s|)^{-\frac{n-1}{2}} \|f\|_{L^1}$$

So the Strichartz pair (p,p) satisfies

$$\frac{1}{p} + \frac{n-2}{2p} = \frac{n-1}{4} \Rightarrow p = \frac{2n}{n-1}$$

Putting it All Together

Now have estimates

$$\|u\|_{L^\infty} \lesssim h^{-\frac{n-1}{2}}$$

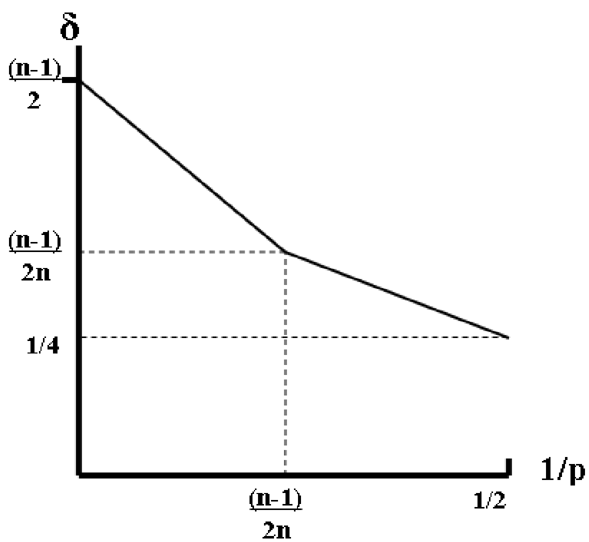
same as over the whole manifold

$$\|u\|_{L^p} \lesssim h^{-\frac{1}{p}} \quad p = \frac{2n}{n-1}$$

from Strichartz estimates

Estimate L^2 directly from $\|W(t)W^\star(s)f\|_{L^2}$ estimate to get $\|u\|_{L^2} \lesssim h^{-\frac{1}{4}}$ Interpolate to get other L^p estimates

General Case



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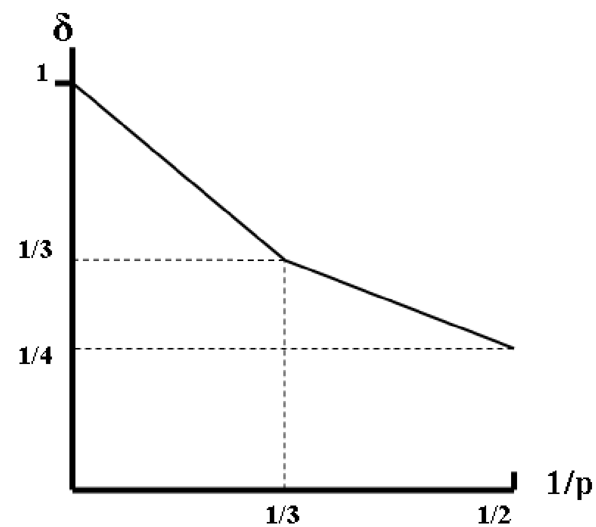

Further Work

- Better estimates in special cases? Toth[5] proved $\|u\|_{L^2(\gamma)} \lesssim \log(\lambda)$ for generic curves in a quantum completely integrable system (2008).
- How does the geometry of the manifold M affect the estimates? Currently working with Hassell on negative curvature, $(\log \lambda)^{1/2}$ improvement.
- Special Hypersurfaces? Burq, Gérard and Tzetkov[1] have $\|u\|_{L^2(\gamma)} \lesssim \lambda^{1/6}$ for γ a curve with positive geodesic curvature. Does this extend to more general regimes?

References

[1] Burq Gérard, Tzetkov, *Restrictions of the laplace-beltrami eigenfunctions to submanifolds*, Duke Math. J., 138(2007), no. 3, 445-486

[2] Keel, Tao, *Endpoint Strichartz estimates, Amer. J. Math. 120 (1998)*, no.5, 955-980

[3] Koch, Tataru, Zworski, *Semiclassical L^p estimates*, Annales Henri Poincaré, 8(2007), no. 5, 885-916

[4] Sogge, *Concerning the L^p norm of spectral clusters for second order elliptic operators on compact manifolds*, J Funct. Anal., 77(1988), 123-134

[5] Toth, *L^2 Restriction bounds for eigenfunctions along curves in the quantum completely integrable case*, preprint (2008)