The method of particular solutions involving tangential derivatives

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Laplacian eigenvalue problem

The Laplacian eigenvalue problem with Dirichlet boundary conditions is to find a nonzero function $\varphi \in C^2(\Omega) \bigcap C(\overline{\Omega})$ and $\lambda > 0$ satisfying

$$-\triangle u = \lambda u \qquad \text{in} \quad \Omega,$$

$$u = 0 \qquad \text{on} \quad \partial\Omega,$$

where Ω be a planar domain and $\triangle$ is the Laplacian operator.

Fox, Henrici, and Moler [4] make famous a "method of particular solutions" (MPS) for computing Laplacian eigenvalues and eigenfunctions. Their idea is as follows

Original method of particular solutions

1. Consider various solutions of $-\Delta u = \lambda u$ in Ω for a given value of λ. These solutions are called particular solutions.

2. Approximate the Laplacian eigenfunctions by the linear combination of the particular solutions $\{\varphi_k(\lambda,\ x)\}_{k=1}^N$

$$\varphi^*(\lambda,\ x) = \sum_{k=1}^{N} c_k\, \varphi_k(\lambda,\ x),$$

3. Try to vary λ until one can choose the suitable c_k such that $\varphi^*(\lambda,\ x)$ vanishes at a number of sample points along the boundary of Ω. Then $\varphi^*(\lambda,\ x)$ is an approximate eigenfunction corresponding to the approximate eigenvalue λ.

Our improvement

We will improve the original MPS with the help of tangential derivatives. Our idea is as follows

Let $\{x_i\}_1^N$ be sampling points on $\partial\Omega$ and $\{\varphi_k(\lambda,\ x)\}_{k=1}^{2N}$ be particular solutions. Denote the tangential derivative of φ_k by $\frac{\partial\varphi_k}{\partial\mathbf{T}}$.

We consider the following system of linear equations

$$\begin{cases} \sum_{k=1}^{2N} d_k^{(N)}\,\varphi_k(\lambda,\ x_i) = 0, & \\ \sum_{k=1}^{2N} d_k^{(N)}\,\frac{\partial\varphi_k}{\partial\mathbf{T}}(\lambda,\ x_i) = 0, & i = 1, ..., N, \end{cases}$$

We search a $\lambda = \lambda_0 > 0$ such that its coefficient determinant vanishes. λ_0 is just the approximate Laplacian eigenvalue.

Replacing λ by λ_0, we find out the nonzero solution $\{\widetilde{d}_k^{(N)}\}_{k=1}^{2N}$ of this system of linear equations. The function

$$\widetilde{\psi}(\lambda_0,\ x) = \sum_{k=1}^{2N} \widetilde{d}_k^{(N)}\, \varphi_k(\lambda_0,\ x)$$

is just an approximate eigenfunction corresponding to the approximate eigenvalue λ_0.

Computation of the tangential derivative

Let L be an arc of the boundary $\partial\Omega$

$$L: \qquad r = \rho(\theta) \quad (\alpha \leq \theta \leq \beta).$$

Then the tangential derivative of the function $f(r, \theta)$ along the curve L at a point (r_i, θ_i) is

$$\frac{\partial f}{\partial \mathbf{T}}(r_i,\ \theta_i) = \left((\rho(\theta_i))^2 + (\rho'(\theta_i))^2\right)^{-\frac{1}{2}}\ \frac{d}{d\theta} f(\rho(\theta),\ \theta)\Big|_{\theta=\theta_i}.$$

Numerical Results

In order to compare the original MPS with our algorithm using tangential derivatives, we will research Laplacian eigenvalue problem in a L-shaped domain

$$\Omega = [-\pi, \pi]^2 - [0, \pi]^2.$$

Its third eigenvalue is $\lambda = 2$ and the corresponding eigenfunction is $u(x, y) = \sin x \sin y$. We will synthesize them using original MPS and our algorithm.

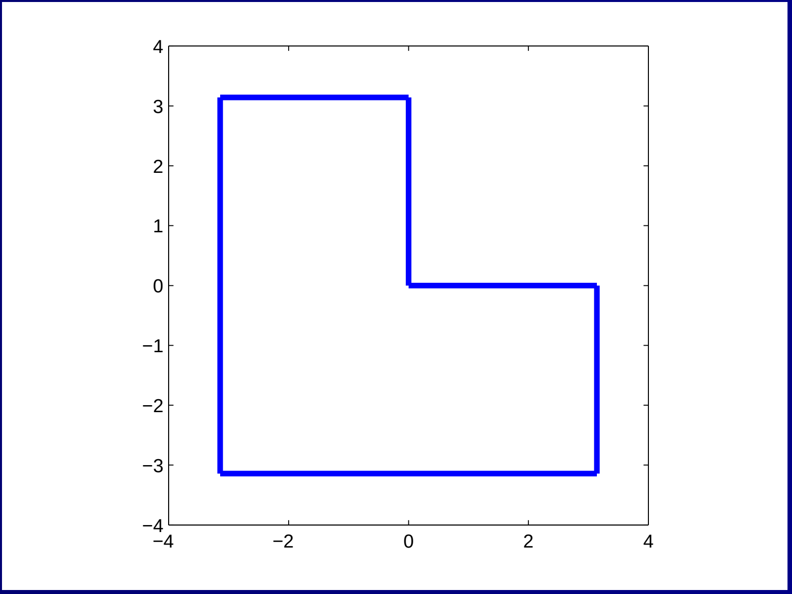

Fig. 1. L-shaped domain

Table 1. The approximation errors of the third eigenvalue

Number of samples	6	8	10	12
Original MPS	0.2810	0.0688	0.0015	0.0011
Our algorithm	0.0067	1.5909×10^{-6}	3.1871×10^{-6}	2.0281×10^{-8}

Number of samples	14	16	18
Original MPS	4.5005×10^{-4}	2.6252×10^{-6}	1.6552×10^{-6}
Our algorithm	7.1809×10^{-13}	1.0534×10^{-12}	8.8818×10^{-16}

Table 2. The approximation errors of the third eigenfunction

Number of samples	6	8	10	12
Original MPS	1.0788	1.1259	0.0168	0.0175
Our algorithm	0.0302	6.1512×10^{-5}	1.7636×10^{-4}	1.4883×10^{-7}

Number of samples	14	16	18
Original MPS	0.0184	4.8813×10^{-5}	6.3697×10^{-5}
Our algorithm	4.4521×10^{-11}	9.3348×10^{-11}	1.6530×10^{-14}

Betcke-Trefethen's improvement of MPS algorithm

In 2005, Betcke and Trefethen [3] improved the original MPS algorithm

Let $\{(r_i, \theta_i)\}_1^{m_B}$ be sample points on the boundary and $\{(r_i', \theta_i')\}_1^{m_I}$ interior points. They researched the matrix

$$A(\lambda) = \begin{pmatrix} A_B(\lambda) \\ A_I(\lambda) \end{pmatrix}$$

where

$$A_B(\lambda) = (\varphi^{(k)}(\lambda,\ r_i,\ \theta_i)), \quad i = 1, ..., m_B,\ k = 1, ..., N,$$
$$A_I(\lambda) = (\varphi^{(k)}(\lambda,\ r_i',\ \theta_i')), \quad i = 1, ..., m_I,\ k = 1, ..., N,$$

and $\{\varphi^{(k)}(\lambda,\ r,\ \theta))\}$ are particular solutions.

Denote QR factorization of the matrix $A(\lambda)$ by $Q(\lambda)$ and $R(\lambda)$, i.e.,

$$A(\lambda) = Q(\lambda)R(\lambda) =: \begin{pmatrix} Q_B(\lambda) \\ Q_I(\lambda) \end{pmatrix} R(\lambda).$$

where Q_B is a $m_B \times N$ matrix and Q_I is a $m_I \times N$ matrix.

Finally, Betcke and Trefethen search λ such that the smallest singular value of $Q_B(\lambda)$ is zero.This suitable λ is just the approximate eigenvalue.

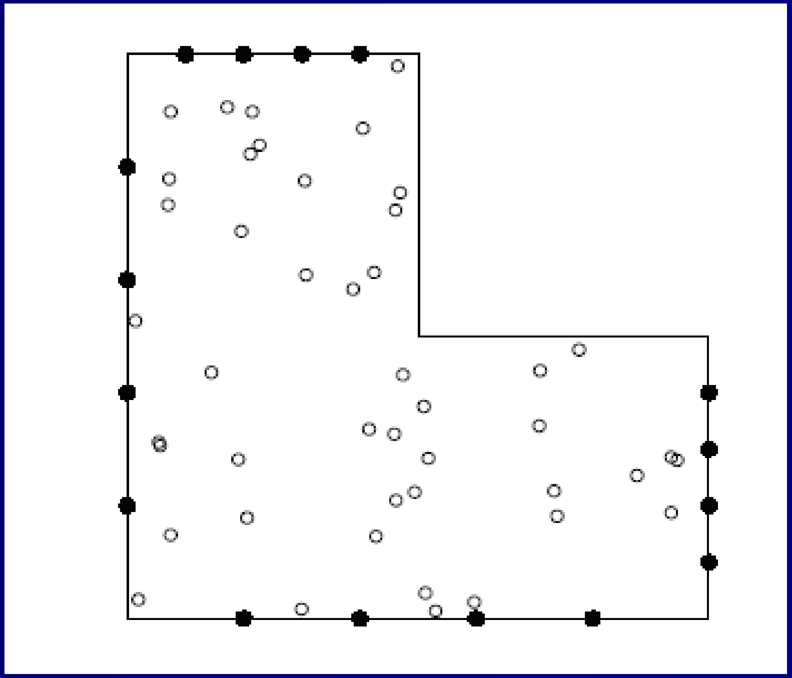

Fig. 2. In Betcke-Trefethen's algorithm, the boundary points are sampled uniformly, and the interior points are chosen randomly.

Our improvement of Betcke-Trefethen's algorithm

Let $\{(r_i, \theta_i)\}_1^{m_B}$ be sample points on the boundary and $\{(r_i', \theta_i')\}_1^{m_I}$ interior points. We research the matrix

$$A(\lambda) = \begin{pmatrix} A_B(\lambda) \\ A_T(\lambda) \\ A_I(\lambda) \end{pmatrix},$$

where $A_B(\lambda) = (\varphi^{(k)}(\lambda,\ r_i,\ \theta_i)),\ i = 1, ..., m_B,\ k = 1, ..., N,$

$$A_T(\lambda) = (\frac{\partial \varphi^{(k)}}{\partial \mathbf{T}}(\lambda,\ r_i,\ \theta_i)),\ i = 1, ..., m_B,\ k = 1, ..., N,$$

$$A_I(\lambda) = (\varphi^{(k)}(\lambda,\ r_i',\ \theta_i')),\ i = 1, ..., m_I,\ k = 1, ..., N.$$

and $\{\varphi^{(k)}(\lambda,\ r,\ \theta))\}$ are particular solutions.

Denote QR factorization of the matrix $A(\lambda)$ by $Q(\lambda)$ and $R(\lambda)$, i.e.,

$$A(\lambda) = Q(\lambda)R(\lambda) =: \begin{pmatrix} Q_B(\lambda) \\ Q_T(\lambda) \\ Q_I(\lambda) \end{pmatrix} R(\lambda).$$

Now we search the suitable λ such that the smallest singular value of

$$\begin{pmatrix} Q_B(\lambda) \\ Q_T(\lambda) \end{pmatrix}$$

is zero. This suitable λ is just the approximate eigenvalue

Numerical Results

In order to compare Betcke and Trefethen's algorithm (BT algorithm) with original MPS, our first algrithm, and our second algorithm, we will compute the first Laplacian eigenvalue in a L-shaped domain

$$\Omega = [-\pi, \pi]^2 - [0, \pi]^2.$$

Our algorithm can obtain the higher accuracy than BT algorithm.

Table 3. The approximation errors of the first eigenvalue

Number of the boundary points	20	30
Original MPS	5.5645×10^{-7}	1.0841×10^{-8}
BT algorithm	5.5645×10^{-7}	1.0841×10^{-8}
Our first algorithm	2.9006×10^{-8}	NA
Our second algorithm	2.9006×10^{-8}	1.0536×10^{-11}

Number of the boundary points	36	52
Original MPS	1.0926×10^{-9}	NA
BT algorithm	1.0926×10^{-9}	6.7671×10^{-12}
Our first algorithm	NA	NA
Our second algorithm	7.9644×10^{-13}	2.0270×10^{-15}

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