Rare Events without Large Deviations.

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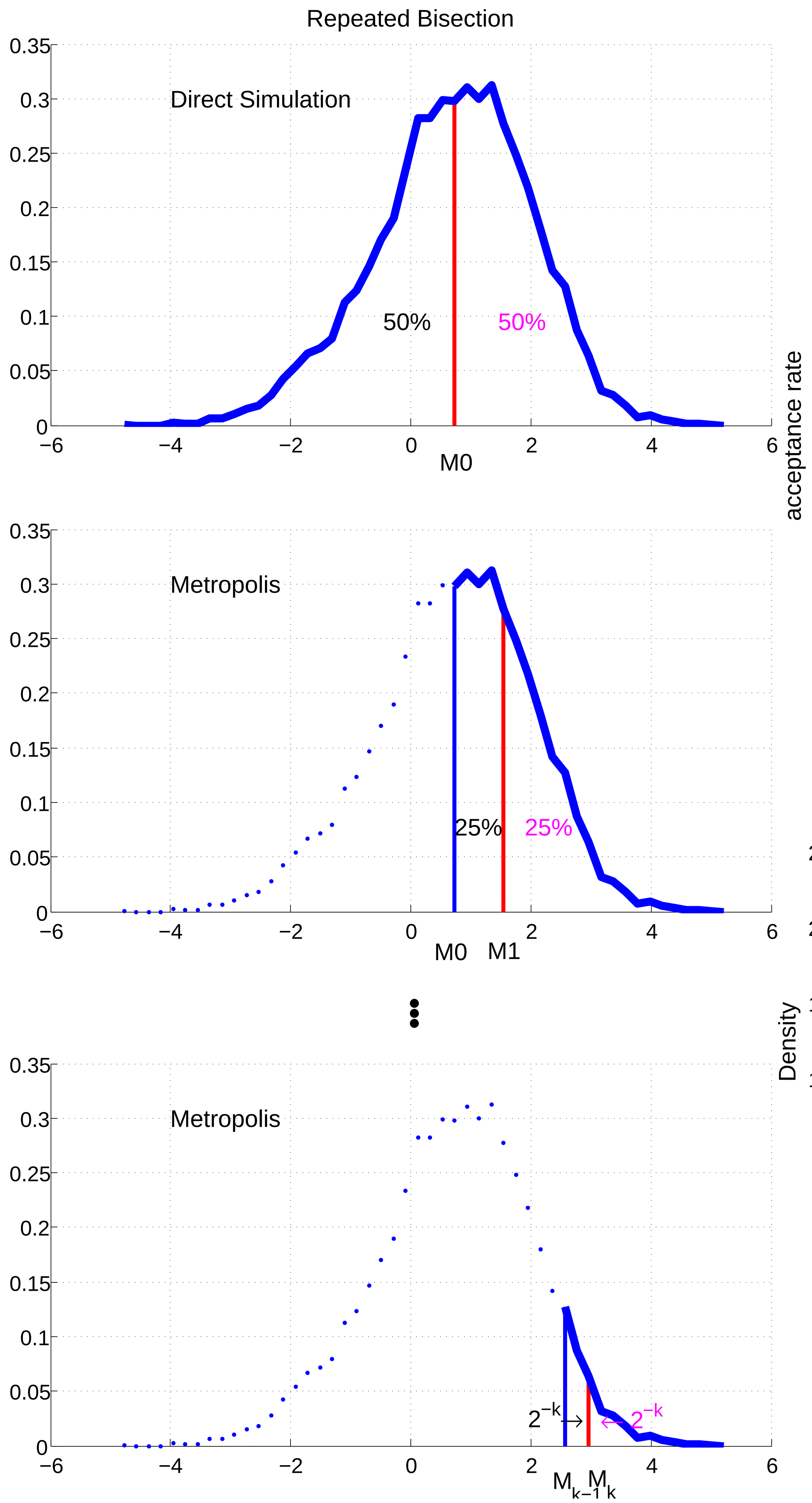
Goal:

- Black box estimates of $P(X > \alpha)$
- No Large Deviation Principle
- No importance sampling
- Yes Metropolis rejection



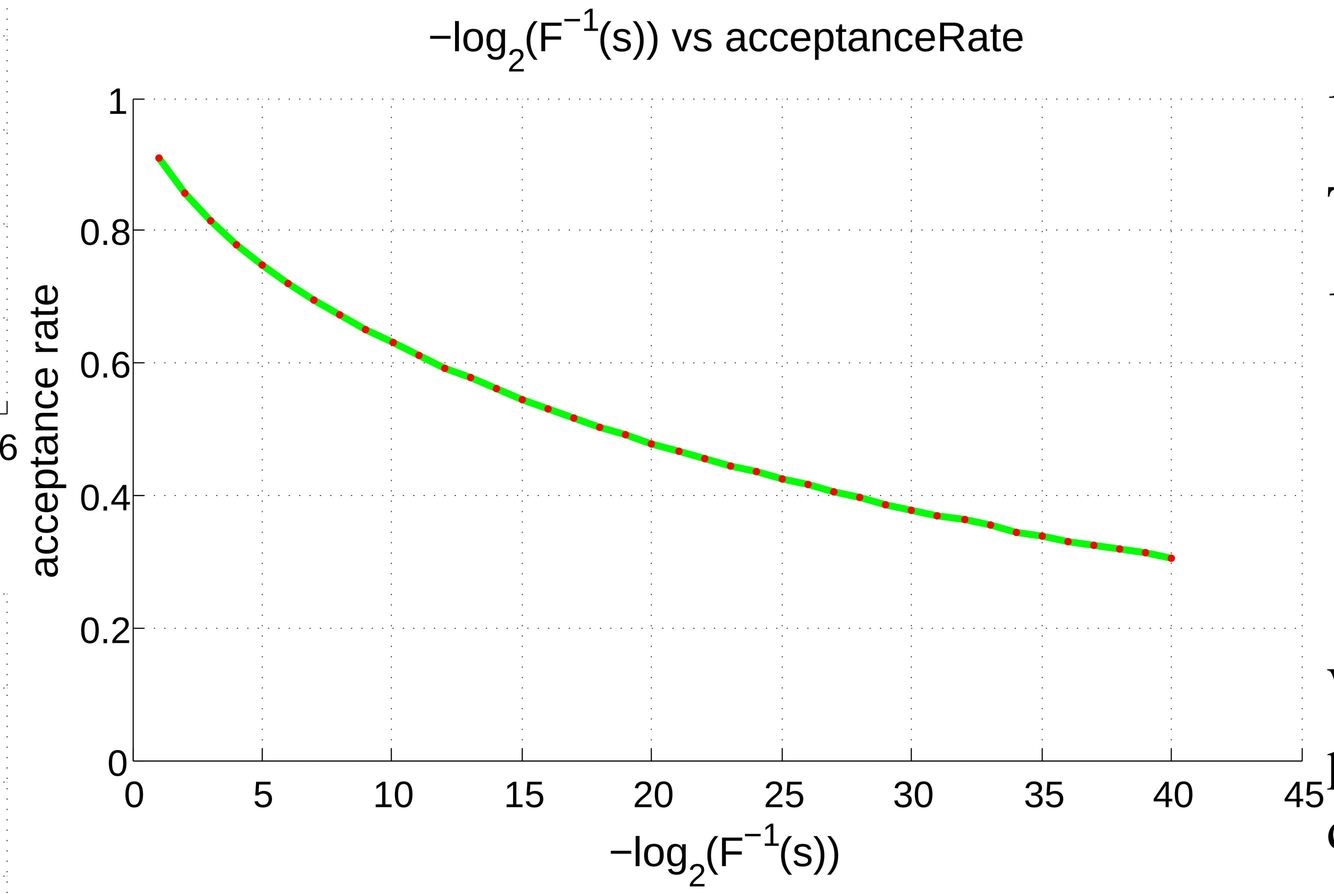
Cramér Test Problem:

$\Pr(Y_1 + \cdots + Y_n \geq n \cdot a)$
$Y_i \sim f_Y(y)$, i.i.d.
Without exponential twist.

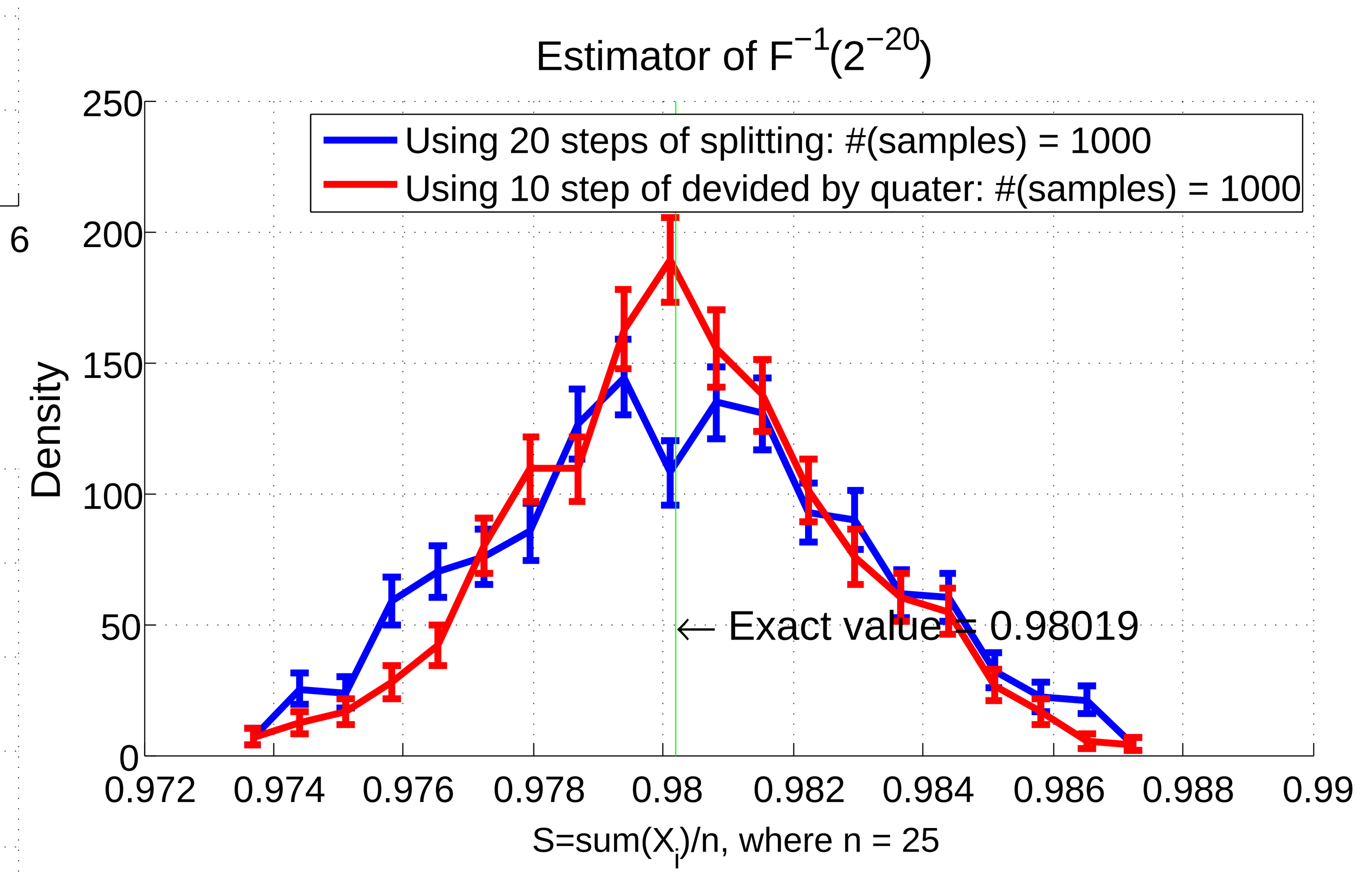
Metropolis Sampling (Cramer)

$Y_1 + \cdots + Y_n \geq n \cdot a$
Sample $\tilde{Y}_i \sim f_Y(y)$
if $Y_1 + \cdots + \tilde{Y}_i + \cdots + Y_n < n \cdot a$ reject

Acceptance rate depending on the quantile



Error estimate of this method



Error Estimate:

Error estimate using median:

Let's assume we make n_s samples at each step, and each sample is independent. Then, if n_s is big enough, the estimation error of the sample median follows a gaussian distribution.

$$F^{-1}(\mathbf{M_0}) \simeq \frac{1}{2} + \frac{Z_1}{2\sqrt{n_s}}$$

$$F^{-1}(\mathbf{M_1}) \simeq \left(\frac{1}{2} + \frac{Z_1}{2\sqrt{n_s}}\right)\left(\frac{1}{2} + \frac{Z_2}{2\sqrt{n_s}}\right) = \frac{1}{4} + \frac{\sqrt{2}Z_3}{2^2\sqrt{n_s}} + O\left(\frac{1}{n_s}\right)$$

$$\vdots$$

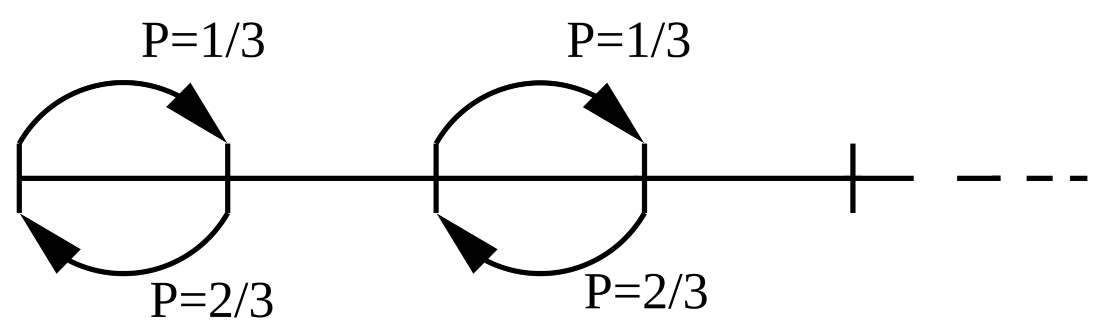
$$F^{-1}(\mathbf{M_{k-1}}) \simeq \frac{1}{2^k} + \frac{\sqrt{k}Z_k}{2^k\sqrt{n_s}} + O\left(\frac{1}{n_s}\right)$$

Thus, the relative error of quantile estimation of $F^{-1}(1/2^k)$ is like below.

$$\frac{F^{-1}(\mathbf{M_{k-1}}) - \frac{1}{2^k}}{\frac{1}{2^k}} \simeq \frac{\sqrt{k}Z_k}{\sqrt{n_s}} + O\left(\frac{1}{n_s}\right)$$

We get the relative error of $\frac{\sqrt{k}Z_k}{\sqrt{n_s}}$ by using $n_s k$ samples. Thus, we can estimate very small probability quantiles with a small number of samples.

Biased Random Walk:



Biased random walk

Let λ be the packet arrival rate
Let μ be the service rate
Let T be the final time
Let $p = \frac{\lambda}{\lambda+\mu}$

Then, the number of packet in the queue at time t, N_t follows below transition.

$$N_0 = 0$$
$$N_{t+1} = \begin{cases} N_t + 1 & \text{with probability p} \\ \max(N_t - 1, 0) & \text{otherwise} \end{cases}$$

Result Graph