The Dirichlet to Neumann Map and Inverse Problems

Gunther Uhlmann

0. An introduction to the electrical impedance tomography problem

1. The Dirichlet problem and the Dirichlet- to Neumann-data map

2. Identification of the boundary values of an isotropic conductivity

3. Layer stripping and boundary determination

4. Construction of exponentially growing voltage potentials

5. Identification of a general isotropic conductivity in the interior, $n \geq 3$

6. Electrical Impedance Tomography, the two dimensional case

7. The identification problem for anisotropic conductors, an introduction

8. The identification problem for anisotropic conductors, the two dimensional case

9. Complex geometrical optics for first order perturbations of the Laplacian

10. Elastic impedance tomography

11. Bibliography

§0. An Introduction to the Electrical Impedance Tomography Problem.

We start by considering a very simple one-dimensional problem, in which we are given an electrically conducting wire of length ℓ. We use x, $0 \leq x \leq \ell$, to refer to an arbitrary point on the wire in terms of its distance from one of the endpoints (fixed). Between any two points $x < y$ on the wire we let $U(x, y)$ denote the voltage difference, given by

$$U(x,y) = u(y) - u(x) \quad , \tag{0.1}$$

Here u denotes the voltage potential, which incidentally only is determined up to a constant. For any $0 \leq x \leq \ell$ we let I_x denote the current passing through the point x (from left to right, say); we subscribe to the convention that a positive current is associated with a voltage drop.

We assume that no interior points of the wire are connected to current sources, and we consider a situation of steady state electric conduction arising from the application of equilibrated sources to the endpoints. Conservation of current now implies that I_x is independent of x. The other main physical principle which we shall need to characterize the voltage potential u is a constitutive law, in this case the socalled Ohm's law:

$$U(x,y) = -I \cdot R(x,y) \quad . \tag{0.2}$$

This law asserts that there is a linear relation between the voltage difference $U(x, y)$ and the constant current I, the positive factor of proportionality being $R(x, y)$, the socalled resistance of the wire between the points x and y. It follows directly from (0.1) and (0.2) that resistance is additive, *i.e.*, $R(x, y) + R(y, z) = R(x, z)$. We shall assume that R is differentiable in some sense. This implies that we may define the resistivity (density) by the relationship

$$R(x,y) = \int_x^y \rho(z)\ dz \quad . \tag{0.3}$$

We assume that ρ is strictly positive (that it is non-negative follows directly from the positivity of $R(x, y)$. It is convenient to introduce the notion of conductivity, γ, by

$$\gamma(x) = \frac{1}{\rho(x)} \quad . \tag{0.4}$$

The identities (0.1)–(0.4) allow us to write

$$u(x) - u(0) = U(0,x) = -I \cdot R(0,x) = -I \int_0^x \rho(y)\ dy = -I \int_0^x \frac{1}{\gamma(y)}\ dy \quad ,$$

or

$$u(x) = u(0) - I \int_0^x \frac{1}{\gamma(y)}\ dy \quad .$$

Differentiation now gives

$$u'(x) = -\frac{I}{\gamma(x)} \quad ,$$

i.e.,

$$\gamma(x)u'(x) = -I \quad .$$

This latter is the infinitessimal version of Ohm's law. An additional differentiation gives

$$(\gamma(x)u'(x))' = 0 \quad . \tag{0.5}$$

We shall base our discussion of the inverse problem on (0.5), not because it is the simplest way to describe the problem in the one dimensional context, but because it is the way that most closely resembles the situation in higher dimensions. At this point we know that all possible steady state voltage potentials are solutions to (0.5). The coefficient $\gamma(x)$ embodies the electric properties of the wire.

Our inverse problem is the following: imagine that the wire is enclosed in a black box, with only the ends exposed. We suppose that we have a device for measuring voltage differences and currents. How much can we learn about the wire (*i.e.*, $\gamma(x)$) by making all possible voltage and current measurements at the two exposed ends? We begin by examining what independent measurements we may perform. First notice that any possible measurement consists of four numbers

$$\{u(0), \gamma(0)u'(0), u(\ell), \gamma(\ell)u'(\ell)\} \quad ;$$

these are the voltages and currents at 0 and ℓ respectively. Notice also that adding a constant to a voltage potential or multiplying a voltage potential by a constant gives another voltage potential. Since a voltage potential (a solution to (0.5)) is uniquely determined by the values of $u(0)$ and $\gamma(0)u'(0)$, it follows that once we have made a measurement corresponding to $u(0) = 0$ and $\gamma(0)u'(0) = 1$, all other measurements can be inferred, and hence can yield no further information about γ. Secondly notice that current is conserved so that in the above case $\gamma(\ell)u'(\ell) = \gamma(0)u'(0) = 1$. The equation (0.5) can be integrated directly to yield

$$u(x) = \gamma(0)u'(0) \int_0^x \frac{dy}{\gamma(y)} + u(0) = \int_0^x \frac{dy}{\gamma(y)} \quad .$$

In other words, all the possible boundary measurements may be inferred from the single measurement

$$\{0, 1, \int_0^1 \frac{dy}{\gamma(y)}, 1\} \quad ,$$

which only contains information about the integral

$$\int_0^1 \frac{dy}{\gamma(y)} \quad ,$$

the total resistance of the wire. In summary, there is only one single experiment to perform: measure the voltage difference corresponding to a unit current passing through the wire. There is only one integral quantity of $\gamma(x)$ that may be determined: the total resistance of the wire.

This is not terribly surprising, nor is it terribly encouraging when it comes to the complete recovery of $\gamma(x)$. The fact is, however, that in higher dimensions it is possible to perform many more experiments and to infer much more information.

Let Ω be a bounded region in $\mathbf{R}^n$ $(n \geq 2)$ which represents a conducting medium. Let $u(x)$ represent a voltage potential (*i.e.*, $u(x) - u(y)$ is the voltage difference measured by a voltmeter with electrodes attached at the points x and y). The current is now represented by a vector which we denote by $i(x)$, and Ohm's law becomes

$$i(x) = -\gamma(x)\nabla u(x) \quad . \tag{0.6}$$

The current is no longer independent of position in Ω, however, since we are considering steady state conduction, charge cannot accumulate in any subset $\tilde{\Omega} \subset \Omega$. This means that the net flow of current across $\partial\tilde{\Omega}$ is zero, *i.e.*,

$$\int_{\partial\tilde{\Omega}} i(x) \cdot \nu(x) \; dS(x) = 0 \quad ,$$

where ν denotes the unit outward normal to $\partial\tilde{\Omega}$. The divergence theorem implies that

$$\int_{\tilde{\Omega}} \nabla \cdot i(x)\ dx = 0 \quad .$$

As $\tilde{\Omega}$ is arbitrary, we have, for every x

$$\nabla \cdot i(x) = 0 \quad .$$

Substituting (0.6) into this identity we arrive at the multidimensional analog of (0.5)

$$\nabla \cdot (\gamma(x)\nabla u(x)) = 0 \quad \text{in } \Omega \quad . \tag{0.7}$$

The coefficient $\gamma(x)$ is in general a positive definite symmetric $n \times n$ matrix; if $\gamma(x)$ is a scalar valued function we say that the medium is isotropic, in all other cases we refer to it as anisotropic.

To give a simple example of the additional information available in dimensions higher than one, consider the following two dimensional example (taken from [K-V I]): Let Ω denote the unit square $\Omega = [0,1]^2$, and let $u(x_1, x_2)$ denote the solution to

$$\begin{aligned} &\nabla \cdot (\gamma(x_1)\nabla u) = 0 \quad \text{in } \Omega, \quad \text{with} \\ &u(0, x_2) = u(1, x_2) = \sin \pi x_2 \quad \text{and} \\ &u(x_1, 0) = u(x_1, 1) = 0 \quad \text{on } \partial\Omega \quad . \end{aligned}$$

As additional measured data we take $\gamma \frac{\partial u}{\partial x_2}|_{x_2=0} = k(x_1)$, $0 < x_1 < 1$. Due to the strong version of the maximum principle it follows immediately that $k(x_1) > 0$ for $0 < x_1 < 1$. Because of the special form of the Dirichlet boundary data we get that that u is of the form

$$u(x_1, x_2) = a(x_1) \sin \pi x_2 \quad ,$$

where $a(x_1)$ is the solution to

$$\begin{aligned} &(\gamma(x_1)a'(x_1))' - \pi^2\gamma(x_1)a(x_1) = 0 \quad \text{in } (0,1) \quad \text{with} \\ &a(0) = a(1) = 1 \quad . \end{aligned} \tag{0.8}$$

The measurement k is related to γ by

$$k(x_1) = \pi\gamma(x_1)a(x_1) \quad \text{or} \quad \gamma(x_1) = \frac{k(x_1)}{\pi a(x_1)} \quad . \tag{0.9}$$

Due to the facts that $k(x_1) > 0$ in $(0,1)$ and $a(0) = a(1) = 1$ we conclude that $0 < a(x_1) < 1$ in $(0,1)$ (and $0 < a(x_1) \leq 1$ on $[0,1]$). A combination of the identities in (0.9) with (0.8) yields the following equation for $b(x_1) = -\log a(x_1) \geq 0$

$$(k(x_1)b'(x_1))' = -\pi^2 k(x_1) \quad \text{in } (0,1) \quad \text{with} \;\; v(0) = v(1) = 0 \;\; ,$$

which by direct integration gives the formula

$$(0.10) \qquad \begin{aligned} v(x_1) &= -\pi^2 \int_0^{x_1} \frac{1}{k(s)} \left[\int_0^s k(t)\, dt - c \right] ds \;, \quad \text{with} \\ c &= \left(\int_0^1 \frac{1}{k(s)}\, ds \right)^{-1} \int_0^1 \frac{1}{k(s)} \int_0^s k(t)\, dt ds \;\; . \end{aligned}$$

From (0.9) it follows that

$$(0.11) \qquad \gamma(x_1) = \frac{k(x_1)}{\pi} e^{b(x_1)} \;\; .$$

The formulas (0.10) and (0.11) together provide the explicit solution to our inverse problem: a formula for $\gamma(x_1)$ in terms of the single measurement $k(x_1)$. For a general γ that depends on both variables x_1 and x_2 a single measurement will not suffice and such an explicit formula is not available. In addition to illustrating the added information available even from a single experiment in two dimensions, the example does illustrate an additional feature of the solution of the inverse problem: the very weak continuous dependence of the reconstruction on the measured data. As is seen even small perturbations of $k(x_1)$ may lead to exponentially large perturbations of the reconstructed γ.

The central aim of impedance tomography is to infer as much as we can about $\gamma(x)$ from multiple boundary measurements of voltages and currents. This is therefore an example of a nondestructive testing situation: it is forbidden to penetrate the interior of Ω with a probe, electrodes may only be attached to the boundary. If Ω is a smooth domain, then the set of all possible smooth measurements consists of

$$(0.12) \qquad \{(f,g) \in C^\infty(\partial\Omega) \times C^\infty(\partial\Omega) \;:\; f = u|_{\partial\Omega},\; g = \gamma \frac{\partial u}{\partial \nu}|_{\partial\Omega} \;\text{ and } u \text{ satisfies (0.7)}\} \;\; .$$

A mathematically (and practically) somewhat more satisfactory approach is to consider the set $\mathcal{C}_q$ of all Cauchy data associated to the equation (0.7). The set $\mathcal{C}_q$ is larger than the set (0.12) – if we restrict attention to solutions of (0.7) with finite energy, then $\mathcal{C}_q$ is the closure of the set (0.12) in the $H^{1/2} \times H^{-1/2}$ norm. Whereas it is natural to think of all the information contained in the set (0.12) (or $\mathcal{C}_q$) as emerging from a special type of experiment – fix voltage pattern and measure current flux across the boundary (or vice versa)– it does also encode the information related to all other possible experiments, such as fixing voltage pattern on part of the boundary, $\partial\Omega_1$, fixing current flux on the remainder of the boundary, $\partial\Omega_2$, and then measuring current flux and voltage pattern on $\partial\Omega_1$ and $\partial\Omega_2$ respectively.

To elaborate a little more on the natural interpretation of (0.12) we mentioned above, consider the Dirichlet problem

$$\nabla \cdot (\gamma \nabla u) = 0 \quad \text{in } \Omega \ , \quad u = f \quad \text{on } \partial\Omega \ . \tag{0.13}$$

This problem is well posed, and therefore the first component of an element of (0.12) can be any function in $C^\infty(\partial\Omega)$. For any such f there is exactly one pair (f, g) contained in (0.12), namely the pair $(f, \gamma\frac{\partial u}{\partial \nu}|_{\partial\Omega})$. We define the map Λ_γ

$$\Lambda_\gamma(f) = \gamma \frac{\partial u}{\partial \nu}|_{\partial\Omega} \quad \text{where } u \text{ solves (0.13)} \ .$$

The map Λ_γ is referred to as the Dirichlet- to Neumann-data map. The set (0.12) is the graph of this map (over $C^\infty(\partial\Omega)$). Our mathematical formulation of the impedance tomography problem is to infer information about γ from the Dirichlet- to Neumann-data map, Λ_γ (or from its inverse, the Neumann- to Dirichlet data map). The use of the Dirichlet- to Neumann-data map among other things permits a simple dimensional analysis which illustrates the relative difficulties of the reconstruction process. The map Λ_γ is a linear map from $C^\infty(\partial\Omega)$ to itself; as such it has a distribution kernel. Roughly speaking this kernel $\lambda_\gamma(x, y)$ permits us to write

$$\Lambda_\gamma f(x) = \int_{\partial\Omega} \lambda_\gamma(x, y) f(y) \ dS(y) \ ,$$

where $dS(y)$ represents surface measure on $\partial\Omega$. We note that if Ω is a bounded domain in $\mathbf{R}^n$ then λ_γ is a "function" of $2 \times (n-1)$ variables, as both x and y belong to the $n-1$

dimensional manifold $\partial\Omega$. If we wish to identify an isotropic conductivity then we must recover a function of n variables, γ. When $n = 1$, λ_γ is a "function" of zero variables – the space of boundary measurements is 1-dimensional – so that formally the problem of reconstructing γ is underdetermined. This is consistent with our earlier analysis. For $n = 2$ the problem is formally determined, since both γ and λ_γ depend on two variables. If $n \geq 3$, the problem is formally overdetermined.

We shall see in chapters 2 through 6 that this purely formal discussion has some relevance. For dimensions $n \geq 2$ we will show that Λ_γ does indeed provide sufficient information to determine an isotropic conductivity (assuming for instance that the conductivity is twice differentiable.) It will also be apparent that the proof of this fact in case $n = 2$ uses much more of the information available through Λ_γ than does the corresponding proof in case $n \geq 3$.

It is not surprising that there are limits to the validity of this purely formal dimensionality argument. As we shall show (in chapter 7) it is not possible to reconstruct an anisotropic conductivity from knowledge of Λ_γ. It seems possible to determine anisotropic γ up to a very natural action by a diffeomorphism. In chapter 8 we shall also give a complete statement and proof of this fact in two dimensions. The dimensionality argument, of course, is not able to distinguish between isotropic and anisotropic conductors, and therefore gives no indication as to the extreme difference between these two cases.

Aside from the identifiability question we shall also study the continuous dependence of the reconstructed conductivity on the boundary data. The continuous dependence results at the boundary are quite good (Lipschitz and Hölder estimates) whereas the continuous dependence results in the interior are quite weak (changes in the measured data may result in exponentially large changes in the interior conductivity).

The application areas of the technique of electrical impedance tomography are diverse – ranging from medical tomography and groundwater seepage to nondestructive inspection of aircraft parts.

§1. The Dirichlet Problem and the Dirichlet- to Neumann-data Map.

In this section we sketch the basic existence and uniqueness proofs for the solution to the Dirichlet Problem

$$L_\gamma u = \nabla \cdot \gamma \nabla u = \frac{\partial}{\partial x_i} \gamma_{ij} \frac{\partial}{\partial x_j} u = F \quad \text{in } \Omega \tag{1.1}$$
$$u = f \quad \text{on } \partial\Omega \ .$$

In doing so we shall use a quite standard Hilbert space approach. We always minimally assume that γ is almost everywhere bounded, symmetric and strictly positive definite, *i.e.*

$$\gamma_{ij} \in L^\infty(\Omega), \quad \gamma_{ij} = \gamma_{ji} \quad \text{and}$$
$$0 < c|\zeta|^2 \le \gamma_{ij}(x)\zeta_i\zeta_j \quad \forall \zeta \in \mathbf{R}^n \quad \text{a.e. } x \in \Omega \ ,$$

the domain Ω is always assumed to be smooth (C^∞). As we proceed we also develop some basic properties of the Dirichlet- to Neumann-data map.

In order to analyze the problem (1.1) it is necessary to recall a few basic properties of the L^2 based Sobolev spaces $H^t(\Omega)$ and $H^t(\partial\Omega)$. On Ω these spaces are for integer exponents $k \ge 0$ defined by

$$H^k(\Omega) = \{u \ : \ \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} u \in L^2(\Omega) \ \ \forall \alpha_j \ge 0 \ \text{ with } \sum \alpha_j \le k\} \ .$$

On $\partial\Omega$ these spaces are for integer exponents defined by means of local charts through the requirement that all derivatives up to a certain order be in L^2. The intermediate spaces for noninteger exponents are in both cases defined by complex interpolation (see [L-M] or [A]). We use the notation $H_0^t(\Omega)$ for the closure of the set $C_0^\infty(\Omega)$ in the $H^t(\Omega)$ norm. The first property we shall need concerns the restriction map $r \ : \ C^\infty(\Omega) \ni u \to u|\partial\Omega \in C^\infty(\partial\Omega)$

Theorem 1.1. *The restriction map R extends to a bounded surjective map*

$$R: \ H^t(\Omega) \to H^{t-\frac{1}{2}}(\partial\Omega) \ \ \textit{for any } t > \frac{1}{2} \ .$$

The kernel of the restriction map R is given by $H_0^1(\Omega) \cap H^t(\Omega)$. Furthermore the map R has a bounded right inverse, i.e., given any $f \in H^{t-\frac{1}{2}}(\partial\Omega)$ there exists $u \in H^t(\Omega)$ such that $Ru = f$, and

$$\|u\|_{H^t(\Omega)} \le C\|f\|_{H^{t-\frac{1}{2}}},$$

the constant C being independent of f.

For a proof of this, see [A]. It is essential here that the exponent t is greater than $\frac{1}{2}$, there is no continuous extension corresponding to $t \le \frac{1}{2}$. The second fact we shall use regarding Sobolev space norms is a version of the socalled Poincaré's inequality

Theorem 1.2. *Let Γ be an open nonempty subset of $\partial\Omega$. There exists a constant $C = C(\Omega, \Gamma)$ such that for any $u \in H^1(\Omega)$*

$$\|u\|^2_{L^2(\Omega)} \le C \left(\sum_{i=1}^{n} \|\frac{\partial u}{\partial x_i}\|^2_{L^2(\Omega)} + \|Ru\|^2_{L^2(\partial\Omega)} \right) \quad .$$

The proof of this estimate may also be found in [A]. The results of Theorem 1.1 and Theorem 1.2 together with the Representation Theorem for Hilbert spaces enable us to prove the unique solvability of the Dirichlet problem (1.1) in an appropriate weak sense. The well known Representation Theorem for Hilbert spaces (frequently referred to as Riesz's Representation Theorem) may be stated as follows

Theorem 1.3. *Let H be a Hilbert space with inner product $(\cdot\ ,\cdot)_H$; then the map*

$$G_H \ : \ H \to H^*$$

from H to its dual space H^*, defined by

$$G_H f := (\cdot\ , f)_H$$

is an isomorphism.

Before we are able to formulate the theorem that asserts the unique solvability of the Dirichlet problem (1.1) we note that $L_\gamma = \frac{\partial}{\partial x_i}\gamma_{ij}\frac{\partial}{\partial x_j}$ may be conveniently viewed as an operator from $H^1(\Omega)$ to the dual of $H^1_0(\Omega)$, denoted $H^{-1}(\Omega)$, by means of the following identity

$$-\langle L_\gamma u, v\rangle := \int_\Omega \gamma_{ij} \frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}\ dx \quad \forall v \in H^1_0(\Omega)\ . \tag{1.2}$$

The notation $\langle\cdot,\cdot\rangle$ is used to signify the standard duality pairing between the Hilbert space $H^1_0(\Omega)$ and its dual, $H^{-1}(\Omega)$. This duality pairing is the extension of the L^2 inner product. If γ is in C^∞, then the above definition is consistent with the natural way in which a differential operator acts on distributions (or in this case functions in $H^1(\Omega)$).

Theorem 1.4. *The mapping*

$$\mathcal{F} \ : \ H^1(\Omega) \to H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \ \ ,$$

defined by

$$\mathcal{F}u := \begin{pmatrix} L_\gamma u \\ Ru \end{pmatrix} \ \ , \tag{1.3}$$

is an isomorphism. That is, for any $F \in H^{-1}(\Omega)$ *and* $f \in H^{\frac{1}{2}}(\partial\Omega)$ *there exists a unique* $u \in H^1(\Omega)$ *such that*

$$\mathcal{F}u = \begin{pmatrix} F \\ f \end{pmatrix} \ \ .$$

This solution u *satisfies the estimate*

$$\|u\|_{H^1(\Omega)} \le C(\|F\|_{H^{-1}(\Omega)} + \|f\|_{H^{\frac{1}{2}}(\partial\Omega)}) \ \ . \tag{1.4}$$

The above theorem guarantees the existence of a unique solution to the boundary value problem in a very specific sense. Because of the definition (1.2) the u which solves $\mathcal{F}u = (F, f)^t$ is also the unique function in $H^1(\Omega)$ which satisfies

$$\begin{gathered} \int_\Omega \gamma_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \ dx = -\langle F, v\rangle \ \ \forall v \in H_0^1(\Omega) \ , \\ \text{and } Ru = f \ \ . \end{gathered} \tag{1.5}$$

This formulation is quite standard; u is frequently referred to as the weak solution of the boundary value problem (1.1).

Proof of Theorem 1.4. We introduce an inner product on $H^1(\Omega)$ by the following formula

$$(u, v)_\gamma := \int_\Omega \gamma_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \ dx + \int_{\partial\Omega} uv \ dS(x) \ \ . \tag{1.6}$$

In the last integral we have for brevity used the notation u and v in place of Ru and Rv; we shall frequently do so whenever no misunderstanding is possible. The notation $\mathcal{H}^1(\Omega)$ refers to the set $H^1(\Omega)$ equipped with this γ-dependent inner product. As $cI \le \gamma \le CI$, it follows from Theorem 1.1 and Theorem 1.2 that the norm, $\|\cdot\|_\gamma$, associated with the inner product $(\cdot\ ,\cdot)_\gamma$ is equivalent to $\|\cdot\|_{H^1(\Omega)}$.

By definition the set $H_0^1(\Omega)$ is closed in $H^1(\Omega)$ and hence it is also closed in $\mathcal{H}^1(\Omega)$. We use the notation $\mathcal{H}_0^1(\Omega)$ to refer to the set $H_0^1(\Omega)$ equipped with the $\gamma-$dependent inner product (1.6). The Hilbert space $\mathcal{H}^1(\Omega)$ may therefore be written as a direct sum of the closed subspace $\mathcal{H}_0^1(\Omega)$ and its (closed) $\gamma-$orthogonal complement $(\mathcal{H}_0^1(\Omega))^{\perp_\gamma}$:

$$\mathcal{H}^1(\Omega) = \mathcal{H}_0^1(\Omega) \bigoplus (\mathcal{H}_0^1(\Omega))^{\perp_\gamma} \quad , \tag{1.7}$$

Since the elements of $\mathcal{H}^1(\Omega)$ and $\mathcal{H}_0^1(\Omega)$ are exactly the same as those of $H^1(\Omega)$ and $H_0^1(\Omega)$ the map L_γ may also be interpreted as a map of $\mathcal{H}^1(\Omega)$ into $H^{-1}(\Omega)$. Using this interpretation we easily see that

$$L_\gamma = G_{\mathcal{H}_0^1} \quad \text{on } \mathcal{H}_0^1(\Omega) \ ,$$

where $G_{\mathcal{H}_0^1}$ denotes the representation isomorphism introduced in Theorem 1.3. In order to see this we have used the fact that the boundary terms of the inner product $(\cdot\, , \cdot)_\gamma$ vanish on $\mathcal{H}_0^1(\Omega)$. The Representation Theorem now implies that L_γ is an isormorphism between $\mathcal{H}_0^1(\Omega)$ and $H^{-1}(\Omega)$. The map L_γ and its inverse are continuous whether the space H^{-1} is equipped with the dual norm coming from $H_0^1(\Omega)$ or that coming from $\mathcal{H}_0^1(\Omega)$.

We next note that the restriction map R, restricted to $(\mathcal{H}_0^1(\Omega))^{\perp_\gamma}$ is an isomorphism onto $H^{\frac{1}{2}}(\partial\Omega)$. This follows directly from Theorem 1.1: the map R maps $\mathcal{H}^1(\Omega)$ continuously onto $H^{\frac{1}{2}}(\partial\Omega)$ and its kernel is exactly $\mathcal{H}_0^1(\Omega)$, it is therefore an isomorphism between the orthogonal complement to its kernel and its range, $H^{\frac{1}{2}}(\partial\Omega)$. The Closed Graph Theorem (or the existence of a bounded right inverse as stated in Theorem 1.1) guarantees the boundedness of $R^{-1} : H^{\frac{1}{2}}(\partial\Omega) \to (\mathcal{H}_0^1(\Omega))^{\perp_\gamma}$. The function $u = R^{-1}f \in (\mathcal{H}_0^1(\Omega))^{\perp_\gamma}$ may also be characterized as the element in $\mathcal{H}^1(\Omega)$ of minimal $\|\cdot\|_\gamma$ norm which satisfies $Ru = f$. According to (1.7) every $u \in \mathcal{H}^1(\Omega)$ has a unique (orthogonal) splitting $u = u_1 + u_2$, with $u_1 \in \mathcal{H}_0^1(\Omega)$ and $u_2 \in (\mathcal{H}_0^1(\Omega))^{\perp_\gamma}$, so that the mapping

$$u \to \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \to \begin{pmatrix} L_\gamma u_1 \\ R u_2 \end{pmatrix}$$

is an isomorphism between $\mathcal{H}^1(\Omega)$ and $H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$. If we now verify that

$$L_\gamma u_1 = L_\gamma u \quad \text{and} \quad R u_2 = R u \quad , \tag{1.8}$$

then it follows that the map $u \to \mathcal{F}u$ is an isomorphism between $\mathcal{H}^1(\Omega)$ and $H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, and from the equivalence of the norms of $\mathcal{H}^1(\Omega)$ and $H^1(\Omega)$ it follows that the map $\mathcal{F}$ is an isomorphims between $H^1(\Omega)$ and $H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, as asserted in the statement of this theorem.

We proceed to verify (1.8). As $u = u_1 + u_2$ and as $Ru_1 = 0$ the identity $Ru_2 = Ru$ is obvious. To verify the statement about L_γ we notice that for any $v \in (\mathcal{H}_0^1(\Omega))^{\perp_\gamma}$ and $w \in \mathcal{H}_0^1(\Omega)$

$$\begin{aligned} 0 = (v, w)_\gamma &= \int_\Omega \gamma_{ij} \frac{\partial v}{\partial x_j} \frac{\partial w}{\partial x_i}\, dx \\ &= -\langle L_\gamma v, w \rangle \quad , \end{aligned}$$

where as before $\langle \cdot\ , \cdot \rangle$ denotes the duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. It follows that $L_\gamma v = 0$ for any $v \in (\mathcal{H}_0^1(\Omega))^{\perp_\gamma}$, and therefore that $L_\gamma u = L_\gamma u_1$. □

We just demonstrated the existence and uniqueness of the weak solution to the Dirichlet problem (1.1). There is another very common characterization of the same weak solution in terms of energy minimization. For this characterization we need the notion of a projection map P: $H^1(\Omega) \to H_0^1(\Omega)$. To be specific we take P to be the orthogonal projection from $H^1(\Omega)$ onto $H_0^1(\Omega)$, say, in the $H^1(\Omega)$ inner product.

Theorem 1.5. *Let F and f be elements of $H^{-1}(\Omega)$ and $H^{\frac{1}{2}}(\Omega)$ respectively. The weak solution to the boundary value problem*

$$L_\gamma u = F \quad in \quad \Omega, \quad u = f \quad on \quad \partial\Omega \ ,$$

introduced by Theorem 1.4 may also be characterized as the unique minimizer to the Dirichlet integral

$$D_F(w) = \frac{1}{2} \int_\Omega \gamma_{ij} \frac{\partial w}{\partial x_j} \frac{\partial w}{\partial x_i}\, dx + < F, Pw > \quad ,$$

in the set $\{w \in H^1(\Omega) \ : \ w|_{\partial\Omega} = f \ \}$.

Proof. Let u denote the solution introduced in Theorem 1.4. As noted this solution satisfies the identity

$$\int_\Omega \gamma_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i}\, dx = -\langle F, v \rangle \quad , \tag{1.9}$$

for any $v \in H_0^1(\Omega)$. It is a simple calculation to check that

$$\frac{1}{2}\int_\Omega \gamma_{ij}\frac{\partial(u-w)}{\partial x_j}\frac{\partial(u-w)}{\partial x_i}\ dx = D_F(w) - D_F(u) + \mathcal{R},$$

for any $w \in H^1(\Omega)$, with the remainder term $\mathcal{R}$ given by

$$\mathcal{R} = \int_\Omega \gamma_{ij}\frac{\partial u}{\partial x_j}\frac{\partial(u-w)}{\partial x_j}\ dx + \langle F, P(u-w)\rangle \quad .$$

For any $w \in H^1(\Omega)$ with $w|_{\partial\Omega} = f$ we have that $u - w \in H_0^1(\Omega)$ and $P(u-w) = u - w$. From (1.9) it now follows that $\mathcal{R} = 0$, *i.e.*,

$$\frac{1}{2}\int_\Omega \gamma_{ij}\frac{\partial(u-w)}{\partial x_j}\frac{\partial(u-w)}{\partial x_i}\ dx = D_F(w) - D_F(u), \tag{1.10}$$

for any $w \in H^1(\Omega)$ with $w|_{\partial\Omega} = f$. Since the left hand side of (1.10) is positive, except when $w = u$, this proves that u is the unique minimizer of $D_F(\cdot)$ in $\{w \in H^1(\Omega)\ :\ w|_{\partial\Omega} = f\ \}$.
□

If the coefficient γ is assumed to be infinitely often differentiable then the weak solution, u, is as regular as the data F and f naturally permit. If the data are sufficiently regular then it follows immediately that the weak solution is also the unique strong solution (by this we mean a function in $C^2(\overline{\Omega})$ which satisfies (1.1) in the classical sense).

Corollary 1.4. *If γ, in addition to being positive definite, is in $C^\infty(\overline{\Omega})$, then the map $\mathcal{F}$ defined by (1.3) is an isomorphism*

$$\mathcal{F}\ :\ H^t(\Omega) \to H^{t-2}(\Omega) \times H^{t-\frac{1}{2}}(\partial\Omega)$$

for any value $t \geq 1$.

The fact that the weak solution to (1.1) satisfies an estimate of the form

$$\|u\|_{H^k(\Omega)} \leq C(\|F\|_{H^{k-2}(\Omega)} + \|f\|_{H^{k-\frac{1}{2}}(\partial\Omega)}) \tag{1.11}$$

for any integer $k \geq 1$ follows from the regularity theory concerning elliptic boundary value problems. We shall not here give a proof of (1.11), but instead refer the reader to chapter

7 of [F]. Corollary 1.4 now follows for arbitrary $t \geq 1$ from interpolation between Sobolev spaces and the estimates (1.11) corresponding to integer k. We note that the interior regularity of u is fairly easy to assert; the regularity up to the boundary is slightly more tricky and in particular requires that the boundary of the domain Ω is C^∞. We also note that the estimate (1.11) for a fixed k does not really require that the conductivity γ be in $C^\infty(\overline{\Omega})$, it suffices that γ be in $C^{k-1}(\overline{\Omega})$.

We are now finally in a position to define the Dirichlet- to Neumann- data map. Consider the boundary value problem (1.1) with F equal to zero

$$\nabla \cdot \gamma \nabla u = 0 \quad \text{in } \Omega, \quad u = f \quad \text{on } \partial\Omega \ .$$

If the boundary data f is in $H^{\frac{3}{2}}(\partial\Omega)$, and γ is in $C^1(\overline{\Omega})$, then the unique solution to this problem, as we have just seen, belongs to $H^2(\Omega)$. Therefore ∇u is in $H^1(\Omega)$ and as a consequence $\nabla u|_{\partial\Omega} = R(\nabla u)$ belongs to $H^{\frac{1}{2}}(\partial\Omega)$. We may now define

$$\Lambda_\gamma f := (\gamma \nabla u) \cdot \nu|_{\partial\Omega)} = \gamma_{ij} \frac{\partial u}{\partial x_j} \nu_i|_{\partial\Omega} \in H^{-\frac{1}{2}}(\partial\Omega) \quad , \tag{1.12}$$

where ν denotes the outward unit normal to $\partial\Omega$. As we shall see below Λ_γ is defined for $f \in H^{1/2}(\partial\Omega)$ (and $\gamma \in L^\infty(\Omega)$) even though the classical formula above, in terms of the restriction map, does not make sense. The classical formula fails to make sense for $f \in H^{1/2}(\partial\Omega)$ because in that case ∇u is generally only in $L^2(\Omega)$ and therefore there is no appropriate notion of a restriction to the boundary. Similarly if γ is only in $L^\infty(\Omega)$ then we would generally only know that $\gamma_{ij} \frac{\partial u}{\partial x_j}$ is in $L^2(\Omega)$ and there would again not be an appropriate notion of its restriction to $\partial\Omega$. In order to define Λ_γ on all of $H^{\frac{1}{2}}(\partial\Omega)$ we shall need its dual space $H^{-\frac{1}{2}}(\partial\Omega)$. The duality pairing between $H^{\frac{1}{2}}(\partial\Omega)$ and $H^{-\frac{1}{2}}(\partial\Omega)$ is the extension of the $L^2(\partial\Omega)$ inner product; we shall also use the notation $\langle \cdot\, , \cdot \rangle$ for this duality pairing.

Theorem 1.6. *Assume that $\gamma \in C^1(\overline{\Omega})$. The Dirichlet to Neumann data map, Λ_γ, defined by (1.12), extends as a bounded map*

$$\Lambda_\gamma \ : \ H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega) \ .$$

Proof. If u, v and γ are arbitrary but smooth functions then Green's formula immediately gives

$$(1.13) \qquad \int_\Omega \gamma_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \, dx = -\int_\Omega \frac{\partial}{\partial x_i} \left(\gamma_{ij} \frac{\partial u}{\partial x_j} \right) \, v \, dx + \int_{\partial\Omega} \gamma_{ij} \frac{\partial u}{\partial x_j} \nu_i \, v \, dS(x) \quad .$$

From the continuity of all the terms involved it is clear that this formula holds for $u \in H^2(\Omega)$, $v \in H^1(\Omega)$ and $\gamma_{ij} \in C^1(\overline{\Omega})$. If u is the solution to $\nabla \cdot \gamma \nabla u = 0$ in Ω, $u = f$ on $\partial\Omega$, defined in the sense of Theorem 1.4, then we know that

$$(1.14) \qquad \int_\Omega \gamma_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \, dx = 0 \quad \forall v \in H_0^1(\Omega) \ .$$

As noted above the solution, u, is in $H^2(\Omega)$ if f is in $H^{\frac{3}{2}}(\partial\Omega)$; it follows from a combination of (1.13) and (1.14) that

$$\int_\Omega \frac{\partial}{\partial x_i} \left(\gamma_{ij} \frac{\partial u}{\partial x_j} \right) \, v \, dx = 0 \quad \forall v \in H_0^1(\Omega) \quad .$$

From the above identity it follows immediately that the L^2 function $\frac{\partial}{\partial x_i} \gamma_{ij} \frac{\partial u}{\partial x_j}$ vanishes. Inserting this fact and the definition of Λ_γ, (1.12), into (1.13) we get

$$(1.15) \qquad \int_{\partial\Omega} \Lambda_\gamma f \, v \, dS(x) = \int_\Omega \gamma_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \, dx \quad \forall v \in H^1(\Omega) \quad .$$

Given any $g \in H^{\frac{1}{2}}(\partial\Omega)$ Theorem 1.1 guarantees the existence of a $v \in H^1(\Omega)$ such that

$$v|_{\partial\Omega} = Rv = g \quad \text{and} \quad \|v\|_{H^1(\Omega)} \leq C \|g\|_{H^{\frac{1}{2}}(\partial\Omega)} \quad .$$

Insertion of this v into (1.15) yields

$$\begin{aligned} \int_{\partial\Omega} \Lambda_\gamma f \, g \, dS(x) &= \int_\Omega \gamma_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \, dx \\ &\leq C \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \\ &\leq C \|f\|_{H^{\frac{1}{2}}(\partial\Omega)} \|g\|_{H^{\frac{1}{2}}(\partial\Omega)} \quad . \end{aligned}$$

Here we have also used the estimate of the $H^1(\Omega)$ norm of u in terms of the $H^{\frac{1}{2}}(\partial\Omega)$ norm of f, as given in Theorem 1.4. Since the duality between $H^{\frac{1}{2}}(\partial\Omega)$ and $H^{-\frac{1}{2}}(\partial\Omega)$ is the extension of the $L^2(\partial\Omega)$ inner product we get, by taking the maximum over all g with $\|g\|_{H^{\frac{1}{2}}(\partial\Omega)} \le 1$,

$$\|\Lambda_\gamma f\|_{H^{-\frac{1}{2}}(\partial\Omega)} \le C\|f\|_{H^{\frac{1}{2}}(\partial\Omega)} \quad .$$

The above estimate was proven under the assumption that $f \in H^{\frac{3}{2}}(\partial\Omega)$. This estimate, however, is sufficient to insure the existence of a unique extension of Λ_γ from $H^{\frac{1}{2}}(\partial\Omega)$ to $H^{-\frac{1}{2}}(\partial\Omega)$. □

Remark 1.1. The extension of Λ_γ, the existence of which we proved above, may easily be expressed in terms of the maps R and G we have already introduced. Recall that the map R is an isomorphism

$$R \ : \ (\mathcal{H}_0^1(\Omega))^{\perp_\gamma} \to H^{\frac{1}{2}}(\partial\Omega) \quad .$$

Its Hilbert space adjoint R^*, defined by

$$(Rv, g)_{H^{\frac{1}{2}}} = (v, R^* g)_\gamma \quad ,$$

is an isomorphism

$$R^* \ : \ H^{\frac{1}{2}}(\partial\Omega) \to (\mathcal{H}_0^1(\Omega))^{\perp_\gamma} \quad .$$

Furthermore, with this notation $(R^*)^{-1} = (R^{-1})^*$, so that

$$((R^*)^{-1}v, g)_{H^{\frac{1}{2}}} = (v, R^{-1}g)_\gamma \quad . \tag{1.16}$$

Following Theorem 1.3 let $G_{\frac{1}{2}}$ denote the representation isomorphism between the space $H^{\frac{1}{2}}(\partial\Omega)$ and its dual, $H^{-\frac{1}{2}}(\partial\Omega)$. We claim that for all $f \in H^{\frac{3}{2}}(\partial\Omega)$

$$\Lambda_\gamma f = G_{\frac{1}{2}}(R^*)^{-1}R^{-1}f - f \quad . \tag{1.17}$$

To see this we calculate

$$\begin{aligned} \langle G_{\frac{1}{2}}(R^*)^{-1}R^{-1}f, g\rangle &= ((R^*)^{-1}R^{-1}f, g)_{H^{\frac{1}{2}}} \\ &= (R^{-1}f, R^{-1}g)_\gamma \quad . \end{aligned} \tag{1.18}$$

Here $\langle \cdot\ , \cdot \rangle$ denotes the duality pairing between $H^{\frac{1}{2}}(\partial\Omega)$ and $H^{-\frac{1}{2}}(\partial\Omega)$. We have used (1.16) to get the last identity. If u and v denote the solutions to $L_\gamma u = L_\gamma v = 0$ with $u|_{\partial\Omega} = f$ and $v|_{\partial\Omega} = g$ respectively, then

$$(R^{-1}f, R^{-1}g)_\gamma = (R^{-1}f, v)_\gamma \quad ,$$

and

$$\int_\Omega \gamma_{ij} \frac{\partial v}{\partial x_j} \frac{\partial R^{-1}f}{\partial x_i}\ dx = \int_\Omega \gamma_{ij} \frac{\partial v}{\partial x_j} \frac{\partial u}{\partial x_i}\ dx \quad .$$

The first of these identities holds because $R^{-1}g - v$ is in $\mathcal{H}_0^1(\Omega)$ and $R^{-1}f$ is in $(\mathcal{H}_0^1(\Omega))^{\perp_\gamma}$. The second holds because $R^{-1}f - u$ is in $H_0^1(\Omega)$ and the function v satisfies $\int_\Omega \gamma \nabla v \nabla w\ dx = 0$ for all $w \in H_0^1(\Omega)$. A combination of the two identities above gives

$$\begin{aligned}(R^{-1}f, R^{-1}g)_\gamma = (R^{-1}f, v)_\gamma &= \int_\Omega \gamma_{ij} \frac{\partial v}{\partial x_j} \frac{\partial R^{-1}f}{\partial x_i}\ dx + \int_{\partial\Omega} fg\ dS(x) \\ &= \int_\Omega \gamma_{ij} \frac{\partial v}{\partial x_j} \frac{\partial u}{\partial x_i}\ dx + \int_{\partial\Omega} fg\ dS(x) \quad .\end{aligned}$$

We may combine this identity with (1.18) to obtain

$$\langle G_{\frac{1}{2}}(R^*)^{-1}R^{-1}f, g\rangle = \int_\Omega \gamma_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i}\ dx + \int_{\partial\Omega} fg\ dS(x) \quad .$$

Using (1.16) we now get that

$$\langle G_{\frac{1}{2}}(R^*)^{-1}R^{-1}f, g\rangle = \int_{\partial\Omega} (\Lambda_\gamma f + f)\ g\ dS(x) = \langle \Lambda_\gamma f + f, g\rangle$$

for any $f \in H^{\frac{3}{2}}(\partial\Omega)$ $g \in H^{\frac{1}{2}}(\partial\Omega)$. The map $f \to G_{\frac{1}{2}}(R^*)^{-1}R^{-1}f - f$ is clearly bounded from $H^{\frac{1}{2}}(\partial\Omega)$ into $H^{-\frac{1}{2}}(\partial\Omega)$, from the last identity we conclude immediately that it is the desired extension of Λ_γ. □

The above discussion leads to a quite general definition of the Dirichlet- to Neumann-data map for any positive definite $\gamma \in L^\infty(\Omega)$. Given f and g in $H^{\frac{1}{2}}(\partial\Omega)$ we let u denote the weak solution to $L_\gamma u = 0$, $u|_{\partial\Omega} = f$ and we let v be any function in $H^1(\Omega)$, with the property that $v|_{\partial\Omega} = g$. We then define $\Lambda_\gamma f \in H^{-\frac{1}{2}}(\partial\Omega)$ by the requirement that

$$\langle \Lambda_\gamma f, g\rangle = \int_\Omega \gamma_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i}\ dx \quad . \tag{1.19}$$

It is easy to see that the right hand side of (1.19) is independent of which v satisfying $v|_{\partial\Omega} = g$ we take. This follows immediately from the fact that

$$\int_\Omega \gamma_{ij} \frac{\partial u}{\partial x_j} \frac{\partial w}{\partial x_i}\ dx = 0 \ \ \forall w \in H_0^1(\Omega) \quad .$$

Furthermore, since there exists $v \in H^1(\Omega)$ with $v|_{\partial\Omega} = g$ and $\|v\|_{H^1(\Omega)} \le C\|g\|_{H^{\frac{1}{2}}(\partial\Omega)}$, the right hand side of (1.19) for fixed u (*i.e.*, for fixed f) defines a bounded linear functional on $H^{\frac{1}{2}}(\partial\Omega)$. This ensures the existence (and uniqueness) of $\Lambda_\gamma f \in H^{-\frac{1}{2}}(\partial\Omega)$ satisfying (1.19). From the inequalities

$$\int_\Omega \gamma_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i}\ dx \le C\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} \le C\|f\|_{H^{\frac{1}{2}}(\partial\Omega)}\|g\|_{H^{\frac{1}{2}}(\partial\Omega)} \quad ,$$

and the definition (1.19) it follows immediately that the "generalized" map Λ_γ is bounded from $H^{\frac{1}{2}}(\partial\Omega)$ to $H^{-\frac{1}{2}}(\partial\Omega)$ It is obvious that this general definition yields exactly the extension of the map defined by (1.12) for $\gamma \in C^1(\overline{\Omega})$.

Returning to the variational characterization of the weak solution to

$$L_\gamma u = 0 \quad \text{in } \Omega, \quad u = f \quad \text{on } \partial\Omega \quad ,$$

we recall that it was shown in Theorem 1.5 that this u is also the unique minimizer to

$$D_0(w) = \frac{1}{2} \int_\Omega \gamma_{ij} \frac{\partial w}{\partial x_j} \frac{\partial w}{\partial x_i}\ dx$$

in the set $\{w \ : \ w \in H^1(\Omega),\ w|_{\partial\Omega} = f\ \}$. The functional Q_γ defined by

$$Q_\gamma(f) = \int_\Omega \gamma_{ij} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i}\ dx = \min_{w \in H^1(\Omega)\ w|_{\partial\Omega} = f} \int_\Omega \gamma_{ij} \frac{\partial w}{\partial x_j} \frac{\partial w}{\partial x_i}\ dx \ ,$$

is a quadratic functional on $H^{\frac{1}{2}}(\partial\Omega)$. From the formula (1.19) it follows immediately that Λ_γ is the selfadjoint linear map associated to this quadratic functional, *i.e.*,

$$\langle \Lambda_\gamma f, f\rangle = Q_\gamma(f) \ \ \forall f \in H^{\frac{1}{2}}(\partial\Omega) \quad .$$

Knowledge of $Q_\gamma(f)$ is therefore the same as knowledge of $\langle \Lambda_\gamma f, f\rangle$. Since knowledge of the two quadratic expressions $\langle \Lambda_\gamma(f+g), (f+g)\rangle$ and $\langle \Lambda_\gamma(f-g), (f-g)\rangle$ by means of the formula

$$4\langle \Lambda_\gamma f, g\rangle = \langle \Lambda_\gamma(f+g), (f+g)\rangle - \langle \Lambda_\gamma(f-g), (f-g)\rangle$$

leads to knowledge of the expression $\langle \Lambda_\gamma f, g\rangle$, it follows that

Proposition 1.7. *Knowledge of $Q_\gamma(f)$ for all $f \in H^{\frac{1}{2}}(\partial\Omega)$ leads to knowledge of $\langle \Lambda_\gamma f, g\rangle$ for all $f, g \in H^{\frac{1}{2}}(\partial\Omega)$ and therefore leads to knowledge of the map Λ_γ. Conversely knowledge of Λ_γ also leads to knowledge of $Q_\gamma(f)$ for all $f \in H^{\frac{1}{2}}(\partial\Omega)$.*

In the later chapters we shall see that many of the developments for the inverse conductivity problem have parallels when the boundary value problem $\nabla \cdot \gamma\nabla u = 0$ in Ω $u = f$ on $\partial\Omega$ is replaced by the Schrödinger equation

$$\begin{aligned} \Delta u + qu = 0 \quad &\text{in } \Omega \\ u = f \quad &\text{on } \partial\Omega \quad . \end{aligned} \tag{1.20}$$

We assume that $q \in L^\infty(\Omega)$. The operator $\Delta + q$ with Dirichlet boundary conditions is self adjoint from $\mathcal{D}(\Delta + q) \subset L^2(\Omega)$ to $L^2(\Omega)$. The domain of definition, $\mathcal{D}(\Delta + q)$, equals $H^2(\Omega) \cap H_0^1(\Omega)$, and the operator has a compact resolvent [K]. Given $f \in H^{\frac{3}{2}}(\partial\Omega)$ the boundary value problem, (1.20), therefore has a unique solution, $u \in H^2(\Omega)$, exactly when zero is not an eigenvalue for $\Delta + q$. In this case it is also possible to show that (1.20) has a unique weak solution for any $f \in H^{\frac{1}{2}}(\partial\Omega)$; this weakly defined solution is in $H^1(\Omega)$. If zero is not an eigenvalue we may define the Dirichlet- to Neumann-data map

$$\Lambda_q f = \frac{\partial u}{\partial \nu}|_{\partial\Omega} \tag{1.21}$$

as a map from $H^{\frac{3}{2}}(\partial\Omega)$ to $H^{\frac{1}{2}}(\partial\Omega)$. Based on what we have seen earlier it is not surprising that this map extends as bounded map from $H^{\frac{1}{2}}(\partial\Omega)$ to $H^{-\frac{1}{2}}(\partial\Omega)$. We summarize the previous discussion with

Theorem 1.8. *Suppose that $q \in L^\infty(\Omega)$ and that zero is not an eigenvalue of $\Delta + q$ with Dirichlet boundary conditions, then the boundary value problem (1.20) has a unique solution satisfying*

$$\|u\|_{H^t(\Omega)} \leq C\|f\|_{H^{t-\frac{1}{2}}(\partial\Omega)} \quad t \geq 1 \quad .$$

The map Λ_q defined by (1.21) has a unique extension as a bounded map

$$\Lambda_q \ : \ H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega) \quad .$$

There is a very direct relation between the isotropic conductivity equation and the Schrödinger equation. Suppose that γ is in $C^2(\overline{\Omega})$ and that $u \in H^2(\Omega)$ is a solution to

$$L_\gamma u = \nabla \cdot \gamma\nabla u = 0 \quad \text{in } \Omega, \quad u = f \quad \text{on } \partial\Omega \quad ,$$

for some $f \in H^{\frac{3}{2}}(\partial\Omega)$. If we define

$$w = \gamma^{\frac{1}{2}} u$$

then we find that

$$\Delta w + qw = 0 \quad \text{in } \Omega \ \ ,$$

with

$$q = -\frac{\Delta \gamma^{1/2}}{\gamma^{1/2}} \quad .$$

At the same time

$$w = \gamma^{\frac{1}{2}} f \quad \text{on } \partial\Omega \ \ ,$$

and we therefore easily calculate that

$$\begin{aligned} \Lambda_q(\gamma^{\frac{1}{2}} f) &= \frac{\partial}{\partial \nu} w|_{\partial\Omega} = \frac{\partial}{\partial \nu}(\gamma^{\frac{1}{2}} w)|_{\partial\Omega} \\ &= \frac{1}{2}\gamma^{-\frac{1}{2}} \frac{\partial \gamma}{\partial \nu} f + \gamma^{-\frac{1}{2}} \Lambda_\gamma f \quad . \end{aligned}$$

Through substitution of $g = \gamma^{\frac{1}{2}} f$ this yields

$$\Lambda_q g = \frac{1}{2}\gamma^{-1} \frac{\partial \gamma}{\partial \nu} g + \gamma^{-\frac{1}{2}} \Lambda_\gamma(\gamma^{-\frac{1}{2}} g) \quad .$$

We have therefore proven

Theorem 1.9. *Let γ be a conductivity in $C^2(\overline{\Omega})$ and define*

$$q = -\frac{\Delta \gamma^{1/2}}{\gamma^{1/2}} \quad ,$$

then

$$\Lambda_q = \gamma^{-\frac{1}{2}} \Lambda_\gamma(\gamma^{-\frac{1}{2}} \ \cdot\ ) + \frac{1}{2}\gamma^{-1} \frac{\partial \gamma}{\partial \nu} I \quad .$$

Remark 1.2. It is interesting to note the similarity between the equation for w in terms of q, and the equation one may write for $\gamma^{1/2}$ in terms of q:

$$\Delta \gamma^{\frac{1}{2}} + q\gamma^{\frac{1}{2}} = 0 \quad .$$

This equation itself will enter in a very natural way in Chapter 6, where we prove the unique identifiability of γ in the two dimensional case. □

We conclude this chapter with a remark owing to the fact that the Dirichlet- to Neumann-data map is not always well defined for a Schrödinger operator. It is therefore often convenient to work with the set of Cauchy data

$$\mathcal{C}_q = \left\{ (f,g) \in H^{\frac{1}{2}}(\partial\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \ : \ f = w|_{\partial\Omega}, \ g = \frac{\partial w}{\partial \nu}|_{\partial\Omega}, \ \text{ with } \Delta w + qw = 0 \right\} \quad .$$

The normal derivative $\frac{\partial w}{\partial \nu}|_{\partial\Omega}$ is here defined by Green's formula:

$$\langle \frac{\partial w}{\partial \nu}|_{\partial\Omega}, \phi \rangle = \int_\Omega \left(\frac{\partial w}{\partial x_i} \frac{\partial v}{\partial x_i} - qwv \right) \, dx \quad ,$$

where $v \in H^1(\Omega)$ satisfies $v|_{\partial\Omega} = \phi$. We note that that when Λ_q exists, $\mathcal{C}_q$ is just its graph

$$\mathcal{C}_q = \{ (f, \Lambda_q f) \ : \ f \in H^{\frac{1}{2}}(\partial\Omega) \ \} \quad .$$

§2. Identification of the Boundary Values of an Isotropic Conductivity.

The goal of this section is to show that if two conductivities γ_1 and γ_2 are in $C^\infty(\bar{\Omega})$ and give rise to the same boundary measurements (i.e., $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$) on the entire boundary, then the conductivities, and their normal derivatives of all orders agree on $\partial\Omega$. This was the first identifiability theorem proved for the conductivity equation $[K - VII]$ and it seems to remain a necessary ingredient in any proof of identifiability in the interior.

We begin our discussion with a simple inequality involving Dirichlet's principle. For any γ_1 and γ_2,

$$Q_{\gamma_1}(\phi) - Q_{\gamma_2}(\phi) = \int_\Omega \gamma_1 |\nabla u_1|^2 \, dx - \int_\Omega \gamma_2 |\nabla u_2|^2 \, dx \tag{2.1}$$

where

$$\begin{array}{ll} L_{\gamma_1} u_1 = 0, & L_{\gamma_2} u_2 = 0, \\ u_1|_{\partial\Omega} = \phi, & u_2|_{\partial\Omega} = \phi. \end{array} \tag{2.2}$$

Recall that Q_γ is the quadratic form associated to the DN map Λ_γ (see Corollary 0.4) and is therefore computable from boundary measurements. As u_2 minimizes the second integral in (2.1)

$$Q_{\gamma_1}(\phi) - Q_{\gamma_2}(\phi) \geq \int_\Omega (\gamma_1 - \gamma_2) |\nabla u_1|^2 \, dx \quad . \tag{2.3}$$

Let $\mathcal{U}_r(p) = \Omega \cap B_r(p)$ denote the intersection of Ω with the ball of radius r about a point $p \in \partial\Omega$. Then

$$Q_{\gamma_1}(\phi) - Q_{\gamma_2}(\phi) \geq \int_{\mathcal{U}_r(p)} (\gamma_1 - \gamma_2) |\nabla u_1|^2 \, dx + \int_{\Omega \setminus \mathcal{U}_r(p)} (\gamma_1 - \gamma_2) |\nabla u_1|^2 \, dx \quad . \tag{2.4}$$

Let $\rho(x)$ denote the distance from x to $\partial\Omega$. The following lemma states that we may construct solutions to (2.2) which have most of their energy concentrated in the neighborhood $\mathcal{U}_r(p)$. Specifically,

Proposition 2.1. *For any integer $\ell \geq 0$, for any $p \in \partial\Omega$, any $r > 0$ and any C, there exists an $\phi \in C^\infty(\partial\Omega)$ such that*

$$\int_{\mathcal{U}_r(p)} \rho^\ell |\nabla u_1|^2 \, dx > C \int_{\Omega \setminus \mathcal{U}_r(p)} |\nabla u_1|^2 \, dx \tag{2.5}$$

As an immediate corollary of Proposition 2.1 we have our main result about boundary identifiability.

Theorem 2.1. *Suppose that γ_1 and γ_2 are in $C^\infty(\bar{\Omega})$ and*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2},$$

then, for any integer $\ell \geq 0$

$$\left(\frac{\partial}{\partial \nu}\right)^\ell \gamma_1 = \left(\frac{\partial}{\partial \nu}\right)^\ell \gamma_2 \qquad \text{on } \partial\Omega \quad . \tag{2.6}$$

Proof. Suppose that (2.6) is false, then, after possibly relabelling γ_1 and γ_2, there exists a point $p \in \partial\Omega$, a neighborhood $\mathcal{U}_r(p)$ and an $\ell \geq 0$ such that

$$\gamma_1 - \gamma_2 \geq c_1 \rho^\ell \qquad \text{in } \mathcal{U}_r(p)$$

for some positive constant c_1. Hence, for all $\phi \in C^\infty(\partial\Omega)$, it follows from (2.4) that

$$\begin{aligned} Q_{\gamma_1}(\phi) - Q_{\gamma_2}(\phi) &\geq c_1 \int_{\mathcal{U}_r(p)} \rho^\ell |\nabla u_1|^2 \, dx - \sup_{\Omega \setminus \mathcal{U}_r(p)} |\gamma_1 - \gamma_2| \cdot \int_{\Omega \setminus \mathcal{U}_r(p)} |\nabla u_1|^2 \, dx \\ &\geq c_1 \left(\int_{\mathcal{U}_r(p)} \rho^\ell |\nabla u_1|^2 \, dx - C_2 \int_{\Omega \setminus \mathcal{U}_r(p)} |\nabla u_1|^2 \, dx \right) \quad . \end{aligned}$$

If we choose ϕ as in Proposition 2.1 (with $C = C_2$) then we have

$$Q_{\gamma_1}(\phi) - Q_{\gamma_2}(\phi) > 0$$

which contradicts the hypothesis $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. □

The assertion of Theorem 2.1 immediately guarantees that all derivatives of γ_1 and γ_2 agree on $\partial\Omega$. As a consequence it follows that Λ_γ uniquely determines γ within the class of real-analytic γ. There is also a local version of Theorem 2.1 which guarantees the coincidendence of all the derivatives of γ_1 and γ_2 near a point p solely based on the coincidence of $\Lambda_{\gamma_1}(f)$ and $\Lambda_{\gamma_2}(f)$ near p for any f with support near p. We now turn to the proof of Proposition 2.1. Our strategy is to estimate the right-hand side of (2.5) from above (Lemma 2.2) , and the left-hand side from below (Lemma 2.3), all in terms of the Dirichlet boundary data ϕ. In a final lemma (Lemma 2.4) we will then construct a particular ϕ by means of which these estimates yield the inequality (2.5).

We begin with a local version of the energy estimates for the Dirichlet problem. We assume throughout that $\gamma \in C^\infty(\bar{\Omega})$.

Lemma 2.1. *Suppose that*

$$L_\gamma u = F$$

$$u|_{\partial\Omega} = f$$

and let $\mathcal{U}_r(p) = \Omega \cap B_r(p)$ be the intersection of Ω with the ball of radius r about $p \in \partial\Omega$. Given any $0 < d < 1$ and any integer $m \geq 1$ there exists C, independent of p and r, such that

$$\|u\|_{H^m\left(\Omega\setminus\mathcal{U}_r(p)\right)} \leq Cr^{-m}\Big\{\|F\|_{H^{m^*-2}\left(\Omega\setminus\mathcal{U}_{dr}(p)\right)} + \|f\|_{H^{m-1/2}\left(\partial\Omega\setminus\mathcal{U}_{dr}(p)\right)} + \|u\|_{L^2\left(\Omega\setminus\mathcal{U}_{dr}(p)\right)}\Big\} , \tag{2.7}$$

with $m^* = \max(m, 2)$.

Proof. Let $0 < \alpha < 1$ and let $\widetilde{\chi}$ denote a smooth cutoff function with

$$\widetilde{\chi}(x) = \begin{cases} 1 & \text{in } \mathbf{R}^n\setminus B_1(0) \\ 0 & \text{in } B_\alpha(0) \end{cases} .$$

For any $s > 0$ define

$$\chi_s(x) = \widetilde{\chi}\left(\frac{x-p}{s}\right)$$

where the dependence on α and p is not explicity indicated.

We first consider the estimate (2.7) in the case $m = 1$. In the above definition of $\widetilde{\chi}$ choose α such that $d < \alpha^2 < 1$ and let w be the solution to

$$L_\gamma w = \chi_{\alpha r} F$$

$$w|_{\partial\Omega} = \chi_{\alpha r} f \quad .$$

We may without loss of generality throughout this proof assume that $r < R_0$, since the left hand side of the inequality (2.7) vanishes for r sufficiently large. From standard energy estimates and the fact that $\|\chi_{\alpha r}\|_{C^1} \leq Cr^{-1}$ we get

$$\begin{aligned} \|w\|_{H^1(\Omega)} &\leq C\big(\|\chi_{\alpha r}F\|_{H^{-1}(\Omega)} + \|\chi_{\alpha r}f\|_{H^{1/2}(\partial\Omega)}\big) \\ &\leq Cr^{-1}\big(\|F\|_{H^{-1}\left(\Omega\setminus\mathcal{U}_{dr}(p)\right)} + \|f\|_{H^{1/2}\left(\partial\Omega\setminus\mathcal{U}_{dr}(p)\right)}\big) \quad . \end{aligned} \tag{2.8}$$

In the last inequality we have used that $\chi_{\alpha r}$ vanishes on a neighborhood of $\mathcal{U}_{dr}(p)$, since $d < \alpha^2$. From classical L^2-estimates [F] we also get

$$\begin{aligned} \|w\|_{L^2(\Omega)} &\leq C(\|\chi_{\alpha r}F\|_{L^2(\Omega)} + \|\chi_{\alpha r}f\|_{L^2(\partial\Omega)}) \\ &\leq C(\|F\|_{L^2\left(\Omega\setminus\mathcal{U}_{dr}(p)\right)} + \|f\|_{L^2\left(\partial\Omega\setminus\mathcal{U}_{dr}(p)\right)}) \quad . \end{aligned} \tag{2.9}$$

The constants C and those that will follow depend on γ, Ω and the particular choice of $\widetilde{\chi}$, but are independent of p and r. Since $\chi_{\alpha r} = 1$ on the support of χ_r the function $u - w$ satisfies

$$\int_\Omega \gamma \nabla(u-w)\nabla(\chi_r^2 v)\ dx = 0 \tag{2.10}$$

for any $v \in H^1(\Omega)$, with $\chi_r^2 v = 0$ on $\partial\Omega$. For the same reason the function $u - w$ satisfies $\chi_r^2(u-w) = 0$ on $\partial\Omega$. By inserting $v = u - w$ into (2.10) and expanding the second differentiation we get

$$\begin{aligned}\int_\Omega \gamma\chi_r^2|\nabla(u-w)|^2\ dx &= -\int_\Omega \gamma\nabla(u-w)2\chi_r\nabla\chi_r(u-w)\ dx\\ &\le C\Big(\int_\Omega \chi_r^2|\nabla(u-w)|^2\ dx\Big)^{1/2}\cdot\Big(\int_\Omega |\nabla\chi_r|^2(u-w)^2\ dx\Big)^{1/2}\\ &\le \frac{1}{2}\min_\Omega\gamma\int_\Omega \chi_r^2|\nabla(u-w)|^2\ dx + Cr^{-2}\|u-w\|^2_{L^2\left(\Omega\setminus\mathcal{U}_{\alpha r}(p)\right)}\quad .\end{aligned}$$

After subtracting the first term in the right hand side from the left hand side and using the fact that $\mathcal{U}_{dr}(p) \subset \mathcal{U}_{\alpha r}(p)$ we get

$$\int_\Omega \chi_r^2|\nabla(u-w)|^2\ dx \le Cr^{-2}\|u-w\|^2_{L^2\left(\Omega\setminus\mathcal{U}_{dr}(p)\right)}\quad ,$$

which since $\chi_r = 1$ on $\Omega\setminus\mathcal{U}_r(p)$ immediately yields

$$\begin{aligned}\|u-w\|^2_{H^1\left(\Omega\setminus\mathcal{U}_r(p)\right)} &= \int_{\Omega\setminus\mathcal{U}_r(p)} |\nabla(u-w)|^2\ dx + \int_{\Omega\setminus\mathcal{U}_r(p)} (u-w)^2\ dx\\ &\le \int_\Omega \chi_r^2|\nabla(u-w)|^2\ dx + \int_{\Omega\setminus\mathcal{U}_r(p)} (u-w)^2\ dx\\ &\le Cr^{-2}\|u-w\|^2_{L^2\left(\Omega\setminus\mathcal{U}_{dr}(p)\right)}\quad ,\end{aligned}$$

or

$$\|u-w\|_{H^1\left(\Omega\setminus\mathcal{U}_r(p)\right)} \le Cr^{-1}\|u-w\|_{L^2\left(\Omega\setminus\mathcal{U}_{dr}(p)\right)}\quad . \tag{2.11}$$

A combination of (2.8), (2.9) and (2.11) yields

$$\begin{aligned}\|u\|_{H^1\left(\Omega\setminus\mathcal{U}_r(p)\right)} &\le \|u-w\|_{H^1\left(\Omega\setminus\mathcal{U}_r(p)\right)} + \|w\|_{H^1(\Omega)}\\ &\le Cr^{-1}\{\|F\|_{H^{-1}\left(\Omega\setminus\mathcal{U}_{dr}(p)\right)} + \|f\|_{H^{1/2}\left(\partial\Omega\setminus\mathcal{U}_{dr}(p)\right)} + \|u-w\|_{L^2\left(\Omega\setminus\mathcal{U}_{dr}(p)\right)}\}\\ &\le Cr^{-1}\{\|F\|_{L^2\left(\Omega\setminus\mathcal{U}_{dr}(p)\right)} + \|f\|_{H^{1/2}\left(\partial\Omega\setminus\mathcal{U}_{dr}(p)\right)} + \|u\|_{L^2\left(\Omega\setminus\mathcal{U}_{dr}(p)\right)}\}\end{aligned}$$

,

which is exactly the desired inequality in case $m = 1$.

We now turn to the case $m \geq 2$. The function $\chi_r u$ satisfies

$$
\begin{aligned}
L_\gamma(\chi_r u) &= \chi_r F + 2\gamma \nabla\chi_r \cdot \nabla u + u L_\gamma \chi_r \\
\chi_r u|_{\partial\Omega} &= \chi_r f \quad .
\end{aligned}
\tag{2.12}
$$

As the right-hand side of (2.12) vanishes inside $\mathcal{U}_{\alpha r}(p)$, the standard elliptic estimate [F] gives

$$
\begin{aligned}
\|\chi_r u\|_{H^m(\Omega)} \leq C\{ &\|\chi_r F\|_{H^{m-2}\left(\Omega\backslash\mathcal{U}_{\alpha r}(p)\right)} \\
&+ \|\chi_r f\|_{H^{m-1/2}\left(\partial\Omega\backslash\mathcal{U}_{\alpha r}(p)\right)} \\
&+ \|\nabla\chi_r \cdot \nabla u + u L_\gamma \chi_r\|_{H^{m-2}\left(\Omega\backslash\mathcal{U}_{\alpha r}(p)\right)} \} \quad ,
\end{aligned}
\tag{2.13}
$$

where the constant C depends on γ and Ω, but is independent of r. We note that, for any set of functions χ and v

$$
\|\chi v\|_{H^m} \leq C \sum_{k=0}^{m} \|\chi\|_{C^k} \|v\|_{H^{m-k}} \quad .
$$

Applying this observation to the last term in (2.13) we get

$$
\|\nabla\chi_r \cdot \nabla u + u L_\gamma \chi_r\|_{H^{m-2}\left(\Omega\backslash\mathcal{U}_{\alpha r}(p)\right)} \leq C \sum_{k=0}^{m-1} \|\chi_r\|_{C^{m-k}} \|u\|_{H^k\left(\Omega\backslash\mathcal{U}_{\alpha r}(p)\right)} .
$$

We note that

$$
\|\chi_r\|_{C^k} \leq C r^{-k}
$$

so that (2.13) gives

$$
\begin{aligned}
\|u\|_{H^m\left(\Omega\backslash\mathcal{U}_r(p)\right)} \leq C r^{-m} \Big\{ &\|F\|_{H^{m-2}\left(\Omega\backslash\mathcal{U}_{\alpha r}(p)\right)} \\
&+ \|f\|_{H^{m-1/2}\left(\partial\Omega\backslash\mathcal{U}_{\alpha r}(p)\right)} \\
&+ \sum_{k=0}^{m-1} r^k \|u\|_{H^k\left(\Omega\backslash\mathcal{U}_{\alpha r}(p)\right)} \Big\} \quad .
\end{aligned}
\tag{2.14}
$$

We may repeat the process which leads to the estimate (2.14) $m-2$ times in order to estimate the norms $\|u\|_{H^k}$, $k = m-1, \ldots, 2$ that appear in the last term of the right hand side. This yields

$$\|u\|_{H^m\left(\Omega\backslash\mathcal{U}_r(p)\right)} \le Cr^{-m}\Big\{\|F\|_{H^{m-2}\left(\Omega\backslash\mathcal{U}_{\alpha^{m-1}r}(p)\right)} \tag{2.15}$$
$$+ \|f\|_{H^{m-1/2}\left(\partial\Omega\backslash\mathcal{U}_{\alpha^{m-1}r}(p)\right)}$$
$$+ \sum_{k=0}^{1} r^k\|u\|_{H^k\left(\Omega\backslash\mathcal{U}_{\alpha^{m-1}r}(p)\right)}\Big\} \ .$$

Using the estimate (2.7) corresponding to $m = 1$, which we have already proven, we may estimate

$$\sum_{k=0}^{1} r^k\|u\|_{H^k\left(\Omega\backslash\mathcal{U}_{\alpha^{m-1}r}(p)\right)} \tag{2.16}$$
$$\le C\{\|F\|_{L^2\left(\Omega\backslash\mathcal{U}_{\alpha^m r}(p)\right)} + \|f\|_{H^{1/2}\left(\partial\Omega\backslash\mathcal{U}_{\alpha^m r}(p)\right)} + \|u\|_{L^2\left(\Omega\backslash\mathcal{U}_{\alpha^m r}(p)\right)}\} \ .$$

A combination of (2.15) and (2.16) yields the desire estimate for $m \ge 2$, provided we select α so that $d < \alpha^m < 1$. □

We next establish

Lemma 2.2. *Suppose that*

$$L_\gamma u = 0$$
$$u|_{\partial\Omega} = \phi \in C^\infty(\partial\Omega)$$

and that

$$\text{supp } \phi \subset \mathcal{U}_{r/2}(p) \cap \partial\Omega$$

then for every integer $1 \le m$, there exists a constant C, independent of p and r, such that

$$\int_{\Omega\backslash\mathcal{U}_r(p)} |\nabla u|^2 \ dx \le Cr^{-m-1}\|\phi\|_{H^{1/2}(\partial\Omega)} \cdot \|\phi\|_{H^{-m+1/2}(\partial\Omega)} \ . \tag{2.17}$$

Proof. Clearly

$$\int_{\Omega\backslash\mathcal{U}_r(p)} |\nabla u|^2 \ dx \le \|u\|^2_{H^1(\Omega\backslash\mathcal{U}_r(p))} \ .$$

In the following let $1/2 < \alpha_2 < \alpha_1 < 1$. Invoking the estimate (2.7) with F identically equal to zero and noting that

$$\text{supp } \phi \subset \mathcal{U}_{r/2}(p) \cap \Omega \ \ ,$$

we get

$$\begin{aligned}\int_{\Omega\setminus\mathcal{U}_r(p)} |\nabla u|^2\ dx &\le Cr^{-2}\|u\|^2_{L^2\left(\Omega\setminus\mathcal{U}_{\alpha_1 r}(p)\right)} \\ &\le Cr^{-2}\int_\Omega \chi u^2\ dx\end{aligned} \tag{2.18}$$

where χ is a smooth positive cutoff equal to one on $\mathbf{R}^n\setminus B_{\alpha_1 r}(p)$ and zero on $B_{\alpha_2 r}(p)$. We denote by w the solution to

$$L_\gamma w = \chi u$$

$$w|_{\partial\Omega} = 0 \quad .$$

We now continue from (2.18) with

$$\begin{aligned}\int_{\Omega\setminus\mathcal{U}_r(p)} |\nabla u|^2\ dx &\le Cr^{-2}\int_\Omega (L_\gamma w)u\ dx \\ &= Cr^{-2}\int_{\partial\Omega}\gamma\frac{\partial w}{\partial\nu}\phi\ ds \\ &\le Cr^{-2}\|\gamma\frac{\partial w}{\partial\nu}\|_{H^{m-1/2}(\partial\Omega)}\cdot\|\phi\|_{H^{-m+1/2}(\partial\Omega)} \\ &\le Cr^{-2}\|w\|_{H^{m+1}(\Omega)}\cdot\|\phi\|_{H^{-m+1/2}(\partial\Omega)} \\ &\le Cr^{-2}\|\chi u\|_{H^{m-1}(\Omega)}\|\phi\|_{H^{-m+1/2}(\partial\Omega)} \\ &\le Cr^{-2}\left(\sum_{k=0}^{m-1}\|\chi\|_{C_k}\|u\|_{H^{m-1-k}\left(\Omega\setminus\mathcal{U}_{\alpha_2 r}(p)\right)}\right)\|\phi\|_{H^{-m+1/2}(\partial\Omega)} \\ &\le C\left(\sum_{k=0}^{m} r^{-k-2}\|u\|_{H^{m-1-k}(\Omega\setminus\mathcal{U}_{\alpha_2 r}(p))}\right)\|\phi\|_{H^{-m+1/2}(\partial\Omega)} \\ &\le C\left(\sum_{k=0}^{m} r^{-k-2}r^{-m+1+k}\|u\|_{L^2\left(\Omega\setminus\mathcal{U}_{r/2}(p)\right)}\right)\|\phi\|_{H^{-m+1/2}(\partial\Omega)} \\ &\le Cr^{-m-1}\|\phi\|_{H^{1/2}(\partial\Omega)}\|\phi\|_{H^{-m+1/2}(\partial\Omega)} \quad .\end{aligned}$$

Notice that Lemma 2.2, with F identically equal to zero was used in deriving the penultimate inequality. There we also used that $\operatorname{supp}\phi\subset\mathcal{U}_{r/2}(p)\cap\partial\Omega$. □

Lemma 2.3. *Let u and ϕ be as in the previous proposition and let $\rho(x)$ denote the distance to $\partial\Omega$. For any integers $\ell\ge 0$, $m\ge 1$ and any real numbers $q>\ell+1$, $s>n/2$, there exist positive constants c and C, independent of p and r, and a constant r_0 depending only on Ω such that*

$$\left(\int_{\mathcal{U}_r(p)}\rho^\ell|\nabla u|^2\ dx\right)^{1/q} \ge \frac{c\|\phi\|_{H^{1/2}(\partial\Omega)}\left(\|\phi\|_{H^{1/2}(\partial\Omega)} - Cr^{-m-1}\|\phi\|_{H^{-m+1/2}(\partial\Omega)}\right)}{\|\phi\|^{2-2/q}_{H^{s+1/2}(\partial\Omega)}} \quad ,$$

$r < r_0$. *If* $\ell = 0$ *then the estimate also holds for* $q = \ell + 1 = 1$.

Proof. We start by considering the case $\ell = 0$. We divide the boundary $\partial\Omega$ into two appropriate nontrivial parts $\partial\Omega_1$ and $\partial\Omega_2$. For $p \in \partial\Omega_1$ there exist a constant, independent of p and r, such that

$$\begin{aligned} \|\phi\|_{H^{1/2}(\partial\Omega)} &\le C\|u\|_{H^1(\Omega)} \\ &\le C\|\nabla u\|_{L^2(\Omega)} \quad . \end{aligned} \tag{2.19}$$

The last estimate follows from Poincaré's inequality, since u vanishes on a fixed subset of $\partial\Omega_2$, independently of p and $r < r_0$ (it is here we need r_0 to be sufficiently small). A similar estimate holds for $p \in \partial\Omega_2$; by taking the larger of the constants C we get that (2.19) holds for all $p \in \partial\Omega$ and $r < r_0$. Therefore

$$\int_{\mathcal{U}_r(p)} |\nabla u|^2 \, dx \ge c\big(\|\phi\|^2_{H^{1/2}(\partial\Omega)} - C\int_{\Omega\setminus\mathcal{U}_r(p)} |\nabla u|^2 \, dx\big) \quad .$$

Employing (H1) we now get get ,

$$\int_{\mathcal{U}_{(p)}} |\nabla u|^2 \, dx \ge c\|\phi\|_{H^{1/2}(\partial\Omega)}\big(\|\phi\|_{H^{1/2}(\partial\Omega)} - Cr^{-m-1}\|\phi\|_{H^{-m+1/2}(\partial\Omega)}\big) \quad , \tag{2.20}$$

as desired.

For $\ell \ge 1$, let a and q' be positive real numbers with $1/q + 1/q' = 1$ and $aq' < 1$. We then have

$$\begin{aligned} \int_{\mathcal{U}_r(p)} |\nabla u|^2 \, dx &= \int_{\mathcal{U}_r(p)} \rho^{-a} \cdot \rho^{a} |\nabla u|^2 \, dx \\ &\le \big(\int_{\mathcal{U}_r(p)} \rho^{-aq'} \, dx\big)^{1/q'} \big(\int_{\mathcal{U}_r(p)} \rho^{aq} |\nabla u|^{2q} \, dx\big)^{1/q} \\ &\le C \cdot \big(\sup_{\mathcal{U}_r(p)} |\nabla u|\big)^{2-2/q} \big(\int_{\mathcal{U}_r(p)} \rho^{aq} |\nabla u|^2 \, dx\big)^{1/q} \\ &\le C\|u\|^{2-2/q}_{H^{s+1}(\Omega)} \big(\int_{\mathcal{U}_r(p)} \rho^{aq} |\nabla u|^2 \, dx\big)^{1/q} \end{aligned}$$

.

For the last inequality we have used the fact that $s > n/2$ together with a well known embedding result for Sobolev spaces [F]. By rearranging the above estimate we get

$$\begin{aligned} \left(\int_{\mathcal{U}_r(p)} \rho^{aq} |\nabla u|^2 \, dx\right)^{1/q} &\ge c\frac{\int_{\mathcal{U}_r(p)} |\nabla u|^2 \, dx}{\|u\|^{2-2/q}_{H^{s+1}(\Omega)}} \\ &\ge c\frac{\int_{\mathcal{U}_r(p)} |\nabla u|^2 \, dx}{\|\phi\|^{2-2/q}_{H^{s+1/2}(\partial\Omega)}} \quad . \end{aligned} \tag{2.21}$$

To justify the previous calculation we needed to insist that

$$1/q + 1/q' = 1 \qquad \text{and} \qquad aq' < 1 \ .$$

Since $q > \ell + 1$ these two requirements are satisfied if we choose

$$q' = \frac{q}{q-1} \qquad \text{and} \qquad a = \frac{\ell}{q}.$$

In that case we furthermore have $aq = \ell$. Insertion of (2.20) into (2.21) now gives

$$\left(\int_{\mathcal{U}_r(p)} \rho^\ell |\nabla u|^2 \, dx\right)^{1/q} \geq \frac{c\|\phi\|_{H^{1/2}(\partial\Omega)}(\|\phi\|_{H^{1/2}(\partial\Omega)} - Cr^{-m-1}\|\phi\|_{H^{-m+1/2}(\partial\Omega)})}{\|\phi\|^{2-2/q}_{H^{s+1/2}(\partial\Omega)}} \ ,$$

as desired. □

The next proposition describes the choice of boundary data we will make to establish Proposition 2.1.

Lemma 2.4. *For any integer $K > 0$ and any $p \in \partial\Omega$, there exists a sequence $\{\phi_N\}_{N=1}^{\infty} \subset C^\infty(\partial\Omega)$ such that*

i) *supp* $\phi_N \searrow \{p\}$

ii) $\|\phi_N\|_{H^{1/2}(\partial\Omega)} = 1$

iii) $\|\phi_N\|_{H^{s+1/2}(\partial\Omega)} \leq C_s N^s \quad \forall s \geq -K$

iv) $\|\phi_N\|_{H^{s+1/2}(\partial\Omega)} \geq c_s N^s \qquad \forall s \geq -K.$

The constants C_s and c_s may be taken independent of p.

Proof. The property iv) is a direct consequence of ii) and iii) and the logarithmic convexity of the H^snorms [A]. As $\frac{1}{2} = \frac{1}{2}(-s+\frac{1}{2}) + \frac{1}{2}(s+\frac{1}{2})$ logaritmic convexity asserts that

$$\|\phi\|_{H^{1/2}(\partial\Omega)} \leq \|\phi\|^{1/2}_{H^{-s+1/2}(\partial\Omega)}\|\phi\|^{1/2}_{H^{s+1/2}(\partial\Omega)} \ .$$

It therefore follows by use of ii) and iii) that

$$1 \leq C_s N^{-s/2}\|\phi_N\|^{1/2}_{H^{s+1/2}(\partial\Omega)}$$

for any $-K \leq s \leq K$, *i.e.*,

$$\|\phi_N\|_{H^{s+1/2}(\partial\Omega)} \geq c_s N^s, \quad \text{for any } \ -K \leq s \leq K \ .$$

For $K < s$ we may for instance use the identity $\frac{1}{2} = \frac{2s}{2s+1} \cdot 0 + \frac{1}{2s+1}(s + \frac{1}{2})$, together with ii) and iii) to derive the estimate

$$\begin{aligned} 1 = \|\phi_N\|_{H^{1/2}(\partial\Omega)} &\leq \|\phi_N\|_{L^2(\partial\Omega)}^{2s/(2s+1)} \|\phi_N\|_{H^{s+1/2}(\partial\Omega)}^{1/(2s+1)} \\ &\leq C_s N^{-s/(2s+1)} \|\phi_N\|_{H^{s+1/2}(\partial\Omega)}^{1/(2s+1)} \quad . \end{aligned}$$

This immediately proves that

$$\|\phi_N\|_{H^{s+1/2}(\partial\Omega)} \geq c_s N^s, \quad \text{for any } K < s \quad .$$

As $\partial\Omega$ is a smooth manifold it suffices, in order to prove this proposition, to produce a sequence of ϕ_N's defined on $\mathbf{R}^{n-1}$ with support shrinking to zero, which satisfies ii) and iii).

Let $\widetilde{\phi}(x) \in C_0^\infty(\mathbf{R}^{n-1})$ be nontrivial and define

$$\widetilde{\phi}_N(x) = \widetilde{\phi}(Nx)$$

(we will eventually choose $\phi_N = \frac{\widetilde{\phi}_N}{\|\widetilde{\phi}_N\|_{H^{1/2}}}$). The functions $\widetilde{\phi}_N$ satisfy

$$\begin{aligned} \|D^\alpha \widetilde{\phi}_N\|_{L^2}^2 &= N^{2|\alpha|} \int_{\mathbf{R}^{n-1}} |D^\alpha \widetilde{\phi}|^2(Nx)\, dx \\ &= N^{2|\alpha|-n+1} \|D^\alpha \widetilde{\phi}\|_{L^2}^2 \end{aligned}$$

so that

$$\|D^\alpha \widetilde{\phi}_N\|_{L^2} \leq C_\alpha N^{|\alpha| - \frac{n-1}{2}} \quad .$$

From this last estimate it follows (again using the logarithmic convexity of the Sobolev norms) that

$$\|\widetilde{\phi}_N\|_{H^s} \leq C_s N^{s - \frac{n-1}{2}} \text{ for } s \geq 0.$$

To obtain the similar estimate for s negative we will use duality, but we must also make a special choice of $\widetilde{\phi}$. Namely, we choose

$$\widetilde{\phi} = D^\alpha \psi(x), \quad |\alpha| = K$$

(i.e., we insist that $\widetilde{\phi}$ is the α-th derivative of a C_0^∞ function ψ for some $|\alpha| = K$). With this choice we may for any C_0^∞ test function η integrate by parts to obtain

$$\begin{aligned}\int_{\mathbf{R}^{n-1}} \widetilde{\phi}_N \eta \; dx &= \int_{\mathbf{R}^{n-1}} (D^\alpha \psi)(Nx)\eta(x) \; dx \\ &= N^{-K} \int_{\mathbf{R}^{n-1}} D^\alpha[\psi(Nx)]\eta(x) \; dx \\ &= (-N)^{-K} \int_{\mathbf{R}^{n-1}} \psi(Nx) D^\alpha \eta(x) \; dx \quad .\end{aligned}$$

For any $0 \le t$ we therefore get

$$\begin{aligned}\left|\int_{\mathbf{R}^{n-1}} \widetilde{\phi}_N \eta \; dx\right| &\le N^{-K} \|\psi(Nx)\|_{H^t} \|D^\alpha \eta\|_{H^{-t}} \\ &\le C_t N^{-K} N^{t - \frac{n-1}{2}} \|\eta\|_{H^{K-t}}\end{aligned}$$

so that

$$\|\widetilde{\phi}_N\|_{t-K} \le C_t N^{t-K-\frac{n-1}{2}} \quad .$$

In terms of $s = t - K$ this gives

$$\|\widetilde{\phi}_N\|_s \le C_s N^{s - \frac{n-1}{2}} \quad \text{for any} \quad -K \le s. \tag{2.22}$$

On the other hand we also know that

$$\|\widetilde{\phi}_N\|_{L^2} = N^{-\frac{n-1}{2}} \|\widetilde{\phi}\|_{L^2} \ge c N^{-\frac{n-1}{2}} \quad ,$$

so that by using logarithmic convexity of the H^s norms and the estimate (2.22) we get

$$\begin{aligned}cN^{-\frac{n-1}{2}} &\le \|\widetilde{\phi}_N\|_{L^2} \\ &\le \|\widetilde{\phi}_N\|_{H^{-1/2}}^{1/2} \|\widetilde{\phi}_N\|_{H^{1/2}}^{1/2} \\ &\le CN^{-\frac{1}{2}(\frac{1}{2} + \frac{n-1}{2})} \|\widetilde{\phi}_N\|_{H^{1/2}}^{1/2}\end{aligned}$$

i.e.,

$$\|\widetilde{\phi}_N\|_{H^{1/2}} \ge c N^{\frac{1}{2} - \frac{n-1}{2}} \quad . \tag{2.23}$$

The estimates (2.22) and (2.23) ensure that

$$\phi_N = \frac{\widetilde{\phi}_N}{\|\widetilde{\phi}_N\|_{H^{1/2}}}$$

satisfy i), ii), and iii). □

We are now in a position to prove Proposition 2.1, and this completes the proof of Theorem 2.1.

Proof of Proposition 2.1. We choose u_1^N satisfying

$$L_{\gamma_1} u_1^N = 0$$

$$u_1^N|_{\partial\Omega} = \phi_N$$

and apply propositions H and L with r fixed and N sufficiently large to see that for any $m \leq K$

$$\int_{\Omega \setminus \mathcal{U}_r(p)} |\nabla u_1^N|^2 \, dx \leq CN^{-m} \ .$$

The constant C depends on r and m. Next, we apply propositions J and L to conclude that for any $q > \ell + 1$, $s > n/2$ and N sufficiently large

$$\left(\int_{\mathcal{U}_r(p)} \rho^\ell |\nabla u_1^N|^2 \, dx \right)^{1/q} \geq \frac{c(1 - CN^{-m})}{N^{s(2-2/q)}} \ ,$$
$$\geq cN^{-s(2-2/q)}$$

or

$$\int_{\mathcal{U}_r(p)} \rho^\ell |\nabla u_1^N|^2 \, dx \geq cN^{-s(2q-2)} \geq cN^{-\ell n - \varepsilon}$$

for any fixed $\varepsilon > 0$. The lemma now follows by selecting

$$m > \ell n$$

(therefore $m > \ell n + \varepsilon$ for ε sufficiently small) and taking N sufficiently large. In order to be able to select such an m we have to choose the integer K from the definition of the sequence $\{\phi_N\}$ larger than ℓn. □

A slightly more careful argument can be used to prove a stability estimate for the inverse problem at the boundary. For that purpose we define the operator norm

$$\|A\|_{1/2,-1/2} = \sup_{\|\phi\|_{H^{1/2}(\partial\Omega)}=1} \|A\phi\|_{H^{-1/2}(\partial\Omega)} .$$

If the operator A is an unbounded self adjoint operator on $L^2(\partial\Omega)$, then

$$\|A\|_{1/2,-1/2} = \sup_{\|\phi\|_{H^{1/2}(\partial\Omega)}=1} |(\phi, A\phi)_{L^2(\partial\Omega)}|$$
$$= \sup_{\|\phi\|_{H^{1/2}(\partial\Omega)}=1} |Q_A(\phi)|$$

Where Q_A denotes the unique quadratic from associated to A [F].

Theorem 2.2. *Suppose that* γ_0 *and* γ_1 *are isotropic* C^∞ *conductivities on* $\bar{\Omega} \subset \mathbf{R}^n$ *satisfying:*

i) $1/E \le \gamma_i \le E$

ii) $\|\gamma_i\|_{C^2(\overline{\Omega})} \le E$,

Given any $0 < \sigma < 1/(n+1)$ *there exists* $C = C(\Omega, E, n, \sigma)$ *such that*

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)} \le C\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{1/2,-1/2} \tag{2.24}$$

and

$$\|\frac{\partial\gamma_1}{\partial\nu} - \frac{\partial\gamma_2}{\partial\nu}\|_{L^\infty(\partial\Omega)} \le C\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|^{\sigma}_{1/2,-1/2} \quad . \tag{2.25}$$

Proof. Let p be an arbitrary point on $\partial\Omega$. In order to verify (2.24) it suffices to prove that

$$|\gamma_1(p) - \gamma_2(p)| \le C\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{1/2,-1/2} \quad ,$$

with C independent of p. If $\gamma_1(p) - \gamma_2(p) = 0$ this estimate is trivial. We may without loss of generality assume that $0 < \gamma_1(p) - \gamma_2(p)$ – if not we just interchange indices. We start from the estimate (2.4) choosing ϕ_N as in Lemma 2.4

$$\begin{aligned}\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{1/2,-1/2} &\ge |Q_1(\phi_N) - Q_2(\phi_N)| \\ &\ge \int_{\mathcal{U}_r(p)} (\gamma_1 - \gamma_2)|\nabla u_1^N|^2\, dx + \int_{\Omega\setminus\mathcal{U}_r(p)} (\gamma_1 - \gamma_2)|\nabla u_1^N|^2\, dx \quad .\end{aligned}$$

Reorganizing this estimate we get

$$\begin{aligned}&\int_{\mathcal{U}_r(p)} (\gamma_1-\gamma_2)|\nabla u_1^N|^2\, dx \\ &\le \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{1/2,-1/2} + \sup_\Omega |\gamma_1 - \gamma_2| \int_{\Omega\setminus\mathcal{U}_r(p)} |\nabla u_1^N|^2\, dx \\ &\le \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{1/2,-1/2} + 2E\cdot Cr^{-m-1}N^{-m} \ , \qquad 0 < m \le K \ ,\end{aligned} \tag{2.26}$$

where the constant C and all those constants that appear in the following are independent of p. To derive the last inequality we have used lemmas 2.2 and 2.4 (ii) and (iii). If we choose r so small that

$$\sup_\Omega |\nabla\gamma_i| \cdot r \le \frac{(\gamma_1 - \gamma_2)(p)}{4} \qquad i = 1, 2 \quad ,$$

then we have

$$(\gamma_1-\gamma_2)(x) = (\gamma_1-\gamma_2)(p) + \int_0^1 \nabla(\gamma_1-\gamma_2)(\tau x+(1-\tau)p)d\tau\cdot(x-p)$$
$$\geq \frac{(\gamma_1-\gamma_2)(p)}{2} \qquad \text{for all } x\in\mathcal{U}_r(p) \quad .$$

The estimate (2.26) therefore yields

$$\frac{(\gamma_1-\gamma_2)(p)}{2}\int_{\mathcal{U}_r(p)}|\nabla u_1^N|^2\ dx \leq \int_{\mathcal{U}_r(p)}(\gamma_1-\gamma_2)|\nabla u_1^N|^2\ dx$$
$$\leq \|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_{1/2,-1/2} + Cr^{-m-1}N^{-m}\ , \qquad 0<m\leq K\ ,$$

and use of lemmas 2.3 (with $\ell = 0$) and 2.4 (ii) and (iii) now gives

$$c(\gamma_1-\gamma_2)(p)(1-Cr^{-m-1}N^{-m}) \leq \|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_{1/2,-1/2} + Cr^{-m-1}N^{-m}\ , \qquad 0<m\leq K\ .$$

If we simply fix m and r and let N approach infinity we obtain

$$c(\gamma_1-\gamma_2)(p) \leq \|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_{1/2,-1/2}$$

with a constant $c>0$, that is independent of p. This completes the proof of (2.24).

We proceed with the verification of (2.25). Since $\|\frac{\partial}{\partial\nu}(\gamma_1-\gamma_2)\|_{L^\infty(\partial\Omega)}$ is uniformly bounded (hypothesis ii)) it clearly suffices to prove (2.25) for $\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_{1/2,-1/2}\leq 1$, in which case

$$\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_{1/2,-1/2}^{1/(n+1)} \leq \|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_{1/2,-1/2}^{\sigma} \leq 1 \quad \text{for any} \quad 0<\sigma<1/(n+1)\ .$$

We may furthermore assume that $\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_{1/2,-1/2}>0$, because if this norm is zero Theorem 2.1 asserts that so is $\|\frac{\partial}{\partial\nu}(\gamma_1-\gamma_2)\|_{L^\infty(\partial\Omega)}$ and (2.25) is trivially satisfied. If $\|\frac{\partial}{\partial\nu}(\gamma_1-\gamma_2)\|_{L^\infty(\partial\Omega)}\leq\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_{1/2,-1/2}^{1/(n+1)}$ the proof of (2.25) is also complete. We may therefore restrict our attention to the case

$$\|\frac{\partial}{\partial\nu}(\gamma_1-\gamma_2)\|_{L^\infty(\partial\Omega)} > \|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_{1/2,-1/2}^{1/(n+1)} > 0 \quad .$$

Let p be a point on $\partial\Omega$ where $|\frac{\partial}{\partial\nu}(\gamma_1-\gamma_2)|$ attains its maximum. To verify (2.25) it suffices to prove

$$|\frac{\partial}{\partial\nu}(\gamma_1-\gamma_2)(p)| \leq C\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_{1/2,-1/2}^{\sigma}\ \ ,$$

with a constant C that is independent of p. By interchange of the role of γ_1 and γ_2 if necessary, we may obtain that $-\frac{\partial}{\partial\nu}(\gamma_1-\gamma_2)(p)>0$. Let $x\in\mathcal{U}_r(p)$ and denote by $q(x)$ the closest point to x on $\partial\Omega$.

$$\begin{aligned}(\gamma_1-\gamma_2)(x) &= (\gamma_1-\gamma_2)(q)+\int_0^1 \nabla(\gamma_1-\gamma_2)\big((1-\tau)q+\tau x\big)d\tau\cdot(x-q)\\ &= (\gamma_1-\gamma_2)(q)-\frac{\partial}{\partial\nu}(\gamma_1-\gamma_2)(q)\rho(x)+O(\rho(x)^2)\\ &= (\gamma_1-\gamma_2)(q)-\frac{\partial}{\partial\nu}(\gamma_1-\gamma_2)(p)\rho(x)+R(x)\rho(x)\end{aligned}$$

where

$$R\le C_1 r\ .$$

Remember, ν is the outward normal to $\partial\Omega$. Choosing $C_1 r=-\frac{1}{2}\frac{\partial}{\partial\nu}(\gamma_1-\gamma_2)(p)$ we now get

$$\begin{aligned}&\int_{\mathcal{U}_r(p)}(\gamma_1-\gamma_2)|\nabla u_1^N|^2\ dx\\ &\qquad\ge\int_{\mathcal{U}_r(p)}(\gamma_1-\gamma_2)(q)|\nabla u_1^N|^2\ dx-\frac{\partial}{\partial\nu}(\gamma_1-\gamma_2)(p)\int_{\mathcal{U}_r(p)}\rho|\nabla u_1^N|^2\ dx\\ &\qquad\qquad -C_1 r\int_{\mathcal{U}_r(p)}\rho|\nabla u_1^N|^2\ dx\\ &\qquad\ge-\sup_{\partial\Omega}|\gamma_1-\gamma_2|\int_{\mathcal{U}_r(p)}|\nabla u_1^N|^2\ dx-\frac{1}{2}\frac{\partial}{\partial\nu}(\gamma_1-\gamma_2)(p)\int_{\mathcal{U}_r(p)}\rho|\nabla u_1^N|^2\ dx\\ &\qquad\ge -C\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_{1/2,-1/2}-\frac{1}{2}\frac{\partial}{\partial\nu}(\gamma_1-\gamma_2)(p)cN^{-n-\varepsilon}\qquad ,0<m\le K\ .\end{aligned}$$

Combining the above estimate with (2.26) and reorganizing we get

$$-\frac{1}{2}\frac{\partial}{\partial\nu}(\gamma_1-\gamma_2)(p)\le C(N^{n+\varepsilon}\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_{1/2,-1/2}+r^{-m-1}N^{n-m+\varepsilon})\ .$$

Since

$$r=-\frac{1}{2C_1}\frac{\partial}{\partial\nu}(\gamma_1-\gamma_2)(p)>\frac{1}{2C_1}\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_{1/2,-1/2}^{1/(n+1)}>0$$

it follows that

$$r^{-m-1}<C\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_{1/2,-1/2}^{-(m+1)/(n+1)}\qquad .$$

If we choose N as follows

$N=$ smallest integer larger than $\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_{1/2,-1/2}^{-\alpha}$ for some $1/(n+1)<\alpha<1/n$

then we obtain

$$
\begin{aligned}
(2.27)\qquad & -\frac{\partial}{\partial\nu}(\gamma_1-\gamma_2)(p) \\
& \leq C(\lambda^{-\alpha(n+\varepsilon)+1}+\lambda^{-(m+1)/(n+1)-\alpha(n-m+\varepsilon)}) \\
& = C(\lambda^{-\alpha(n+\varepsilon)+1}+\lambda^{(\alpha-1/(n+1))m-1/(n+1)-\alpha(n+\varepsilon)}) \qquad ,0<m\leq K \ ,
\end{aligned}
$$

with $\lambda=\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_{1/2,-1/2}\leq 1$. We now select

$$
\alpha=\frac{1-\sigma'}{n} \qquad \text{with} \qquad \sigma'=\frac{n\sigma+\varepsilon}{n+\varepsilon} \quad .
$$

Since $0<\sigma<1/(n+1)$ we also have $0<\sigma'<1/(n+1)$ for ε sufficiently small, and therefore the above choice of α satisties $1/(n+1)<\alpha<1/n$. With this choice of α we also have $-\alpha(n+\varepsilon)+1=\sigma$. We now choose m (and K) sufficiently large that

$$
(\alpha-\frac{1}{n+1})m-\frac{1}{n+1}-\alpha(n+\varepsilon)>\sigma \quad .
$$

After insertion into (2.27) this yields

$$
\|\frac{\partial}{\partial\nu}(\gamma_1-\gamma_2)\|_{L^\infty(\partial\Omega)}=-\frac{\partial}{\partial\nu}(\gamma_1-\gamma_2)(p)\leq C\lambda^\sigma=C\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|^\sigma_{1/2,-1/2} \quad ,
$$

as desired. □

§3. Layer stripping and boundary determination

A Boundary determination

In this section we give an alternative approach to the proof of Theorem 2.1 developed first in [S-U I] which uses the fact that Λ_γ is a pseudodifferential operator of order 1. Then one computes its full symbol in appropriate coordinates. In [L-U] this approach was further simplified by using a "factorization" method. We will follow this approach here because it also leads to a Riccati type equation satisfied by the DN map. This, and the boundary determination of the conductivity, are the key elements of the layer stripping algorithm developed in [S-C-I]. Also the factorization method leads to boundary determination results for more general equations and systems ([L-U], [N-S-U], [N-U I]). Furthermore it provides an easy way to show that the DN map is a pseudodifferential operator. For related approaches using the singularities of the Green's kernel instead see [A] and [N I].

We start with a very simple example. Let $\Omega = \mathbb{R}^n_+ = \{x = (x', x_n), x_n > 0\}$. Then $\partial\Omega = \mathbb{R}^{n-1}$. Let $f \in H^{\frac{1}{2}}(\mathbb{R}^{n-1})$. Let us consider the unique solution, $u \in H^1(\mathbb{R}^n_+)$, of

$$\Delta u = 0 \text{ in } \mathbb{R}^n_+ \tag{3.1}$$
$$u|_{\partial\Omega} = f$$

Then the DN map is

$$f \to -\left.\frac{\partial u}{\partial x_n}\right|_{\mathbb{R}^{n-1}} \tag{3.2}$$

where u solves (3.1).

We factorize

$$-\Delta = (D_{x_n} + i\sqrt{-\Delta'})(D_{x_n} - i\sqrt{-\Delta'}) \tag{3.3}$$

where $D_{x_j} = \frac{1}{i}\frac{\partial}{\partial x_j}$, $j = 1, \ldots, n$ and $-\Delta' = -\sum_{j=1}^{n-1}\frac{\partial^2}{\partial x_j^2}$. $\sqrt{-\Delta'}$ is the pseudodifferential operator given by

$$\sqrt{-\Delta'} f = \frac{1}{(2\pi)^{n-1}}\int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'}|\xi'|\widehat{f}(\xi')d\xi'.$$

The point is that we can solve

$$(3.4) \qquad (D_{x_n} + i\sqrt{-\Delta'})u = 0 \text{ in } \mathbb{R}^n_+$$
$$u\big|_{x^n=0} = f.$$

We simply take for $x_n > 0$

$$(3.5) \qquad u(x', x_n) = \int e^{ix'\xi'} e^{-x_n|\xi'|} \widehat{f}(\xi') d\xi'.$$

From (3.4) we then deduce that

$$-\left.\frac{\partial u}{\partial x_n}\right|_{x_n=0} = \sqrt{-\Delta'} f.$$

So the DN map in this case is just $\sqrt{-\Delta'}$ whose full symbol is $|\xi'|$. Note that the term $(D_{x_n} - i\sqrt{-\Delta'})$ behaves like a heat equation in $\mathbb{R}^n_+$ and $(D_{x_n} + i\sqrt{-\Delta'})$ behaves like a backwards heat equation.

Now we try a similar idea for $(-\Delta + q)$, $q \in C^\infty(\overline{\Omega})$ where Ω is a bounded domain with smooth boundary. First we take coordinates near a point $x_0 \in \partial\Omega$ so that locally $\Omega = \{(x', x_n), x_n > 0\}$ and $-\left.\frac{\partial}{\partial x_n}\right|_{\partial\Omega} = \left.\frac{\partial}{\partial \nu}\right|_{\partial\Omega}$ with ν the unit outer normal to $\partial\Omega$.

In these coordinates

$$(3.6) \qquad -\Delta + q = (D^2_{x_n} + iE(x)D_{x_n} + Q(x, D_{x'}) + q)$$

with $E \in C^\infty(\overline{\Omega})$ real-valued and $Q(x, D_{x'})$ a differential operator of order 2 in x', with no zero order term, depending smoothly on x_n, with full symbol $g_2(x, \xi') + g_1(x, \xi')$ with $g_2 > 0$ and g_i homogeneous of degree i in ξ', $i = 1, 2$.

We try to find an operator $B(x, D_{x'})$ so that we have the factorization

$$(3.7) \qquad (-\Delta + q) = (D_{x_n} + iE(x) + iB(x, D_{x'}))(D_{x_n} - iB(x, D_{x'}).$$

Using (3.6) and (3.7) $B(x, D_{x'})$ must solve

$$(3.8) \qquad i[D_{x_n}, B(x, D_{x'})] + EB(x, D_{x'}) + B^2 - Q - q = 0$$

where $[A, B] = AB - BA$ denotes the commutator. Notice that (3.8) is a Riccati type equation for B. We solve (3.8) using the calculus of pseudodifferential operators. If $b(x, \xi')$ denotes the full symbol of $B(x, D_{x'})$ a pseudodifferential operator of order 1 then the full symbol of $i[D_{x_n}, B(x, D_{x'}]$ is $\frac{\partial}{\partial x_n} b(x, \xi')$. The full symbol of B^2 is $\sum_\alpha \frac{1}{\alpha!} \partial^\alpha_{\xi'} b(x, \xi') D^\alpha_{x'} b(x, \xi')$ where the $\sum$ is interpreted asymptotically (as usual).

The full symbol of EB is $b(x, \xi')E(x')$. Therefore the equation we must solve for $b(x, \xi')$ is

$$\partial_{x_n} b(x, \xi') + b(x, \xi')E(x) + \sum_\alpha \frac{1}{\alpha!} \partial^\alpha_{\xi'} b(x, \xi') D^\alpha_{x'} b(x, \xi') - g_2(x, \xi') - g_1(x, \xi') - q = 0. \tag{3.9}$$

Now we write

$$b(x, \xi') \sim \sum_{j \leq 1} b_j(x, \xi') \tag{3.10}$$

with b_j homogeneous of degree j in ξ'.

Now we compare terms of the same homogeneity in (3.9). The term homogeneous of degree 2 in (3.9) is

$$b_1^2(x, \xi') - g_2(x, \xi') = 0.$$

Therefore we choose

$$b_1(x, \xi') = \sqrt{g_2(x, \xi')}. \tag{3.11}$$

We choose the positive sign in the square root in (3.11) since we want the term $D_{x_n} - iB(x, D_{x'})$ to behave like a heat equation in Ω. The term homogeneous of degree 1 in (3.9) is

$$\partial_{x_n} b_1(x, \xi') + b_1(x, \xi')E(x) + \sum_{j=1}^{n-1} \partial_{\xi'_j} b_1 D_{x'_j} b_1 + 2b_0 b_1 - g_1 = 0.$$

Therefore we choose

$$b_0 = \frac{1}{2b_1} \left\{ -\partial_{x_n} b_1(x, \xi') - b_1(x, \xi')E(x) - \sum_{j=1}^{n-1} \partial_{\xi'_j} b_1 D_{x'_j} b_1 + g_1 \right\}. \tag{3.12}$$

We will do one more step. The term homogeneous of degree 0 in (3.9) is

$$\partial_{x_n} b_0(x,\xi') + b_0(x,\xi')E(x) + 2b_{-1}b_1 + b_0^2$$
$$+\sum_{|\alpha|=1} \partial_{\xi'}^\alpha b_0 D_{x'}^\alpha b_1 + \sum_{|\alpha|=1} \partial_{\xi'}^\alpha b_1 D_{x'}^\alpha b_0 + \sum_{|\alpha|=2} \frac{1}{\alpha!}\partial_{\xi'}^\alpha b_1 D_{x'}^\alpha b_1 - q = 0$$

We then choose

$$b_{-1} = \frac{1}{2b_1}\Big\{-\partial_{x_n} b_0(x,\xi') - b_0(x,\xi')E(x) - \sum_{|\alpha|=1} \partial_{\xi'}^\alpha b_0 D_{x'}^\alpha b_1 - \sum_{|\alpha|=1} \partial_{\xi'}^\alpha b_1 D_{x'}^\alpha b_0 - \sum_{|\alpha|=2} \frac{1}{\alpha!}\partial_{\xi'}^\alpha b_1 D_{x'}^\alpha b_1 + q\Big\}. \tag{3.13}$$

Now the inductive procedure is clear. For any $j < -2$, collecting terms homogeneous of degree $j+1$, we obtain

$$b_j = \frac{1}{2b_1}\Big\{-\partial_{x_n} b_{j+1} - b_{j+1}E - 2\sum_{\substack{l+k=j+1\\ l,k\geq j+1}} b_l b_k - \sum_{\substack{|\alpha|\geq 1\\ l+k-|\alpha|=j+1\\ l,k\leq 1}} \frac{1}{\alpha!}\partial_{\xi'}^\alpha (b_l) D_{x'}^\alpha (b_k)\Big\} \tag{3.14}$$

Note, that this forces $l, k \geq j + |\alpha| \geq j+1$, i.e. the procedure is recursive. Then we have proven

Theorem 3.1. *In local coordinates (x', x_n) as chosen above, there exists a pseudodifferential operator $B(x, D_{x'})$ of order 1 depending smoothly on x_n such that*

$$-\Delta + q = (D_{x_n} + iE(x) + iB(x, D_{x'}))(D_{x_n} - iB(x, D_{x'})) \tag{3.15}$$

modulo a smoothing operator.

Remark. The equation is solved modulo smoothing since we have only compared the full symbol of both sides of (3.15).

Now we can solve the pseudodifferential equation

$$(D_{x_n} - iB(x, D_{x'}))u = 0 \qquad \text{(mod smoothing)} \tag{3.16}$$
$$u|_{\partial\Omega} = f$$

in the form

$$u(x', x_n) = \int e^{ix'\xi'} e^{-x_n|\xi'|} a(x,\xi') \widehat{f}(\xi') d\xi' \tag{3.17}$$

with $a \sim \sum_{j \leq 0} a_j$, $a_j(x, \xi')$ homogeneous of degree j in ξ'. (See [T].)

Now the other term in the factorization (3.11) is a smoothing operator (see [T]). Therefore we conclude that if u is the solution of

$$(3.18) \qquad (-\Delta + q)u = 0$$
$$u|_{\partial\Omega} = f.$$

Then in local coordinates (x', x^n)

$$D_{x_n} u = iB(x, D_{x'})u \qquad \text{mod smoothing.}$$

Therefore

$$(3.19) \qquad \Lambda_q = B(x', 0, D_{x'}) \qquad \text{mod smoothing.}$$

proving that the DN map is a pseudodifferential operator of order 1 on $\partial\Omega$. Now we prove

Theorem 3.2. *From the full symbol of Λ_q we can recover $\partial^\alpha q|_{\partial\Omega} \, \forall \, \alpha$.*

Proof. Using (3.19) we need only to compute the full symbol of $B(x', 0, D_{x'})$ i.e. $b(x', 0, \xi') \sim \sum_{i \leq 1} b_j(x', 0, \xi')$.

The terms b_1, b_0 don't give any information on q (see (3.11) and (3.12)). Now from (3.13) we conclude that if we know $b_{-1}(x', 0, \xi')$ we can determine $q(x', 0)$ since all of the other terms in the RHS of (3.13) are known.

Proceeding inductively: if we know that from b_{-j+1} we can determine $\frac{\partial^j q}{\partial x_n^j}(x', 0)$, and if we know b_k, $k \geq -j+1$, then from (3.14) we conclude that we can recover from $b_{-j}(x', 0, \xi')$, $\frac{\partial^{j+1} q}{\partial x_n^{j+1}}(x', 0)$ finishing the proof.

We now use Theorem 2 to prove the Kohn-Vogelius result.

Theorem 3.3. *Let $\gamma_i \in C^\infty(\overline{\Omega})$, $\gamma_i \geq \epsilon > 0$, so that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. Then*

$$\partial^\alpha \, \gamma_1|_{\partial\Omega} = \partial^\alpha \, \gamma_2|_{\partial\Omega} \qquad \forall \, \alpha$$

Remark. This result is also local, i.e., one only needs to take $x_0 \in \partial\Omega$ and a neighborhood $U(x_0)$ of x_0 in Ω so that $\Lambda_{\gamma_1}(f)|_{U(x_0)} = \Lambda_{\gamma_2}(f)|_{U(x_0)} \, \forall \, f$ supported in $U(x_0)$ to conclude that $\partial^\alpha \gamma_1(x_0) = \partial^\alpha \gamma_2(x_0) \, \forall \, \alpha$.

We recall Theorem 1.9 relating Λ_{q_j} and Λ_{γ_j} if $q_j = \frac{\Delta\sqrt{\gamma_j}}{\sqrt{\gamma_j}}$, $j = 1, 2$

$$\Lambda_{q_j} f = \gamma_j^{-\frac{1}{2}}\Big|_{\partial\Omega} \Lambda_{\gamma_j} \left(\gamma_j^{-\frac{1}{2}}\Big|_{\partial\Omega} f\right) + \frac{1}{2}\left(\gamma_j^{-1}\frac{\partial\gamma_j}{\partial\nu}\right)\Big|_{\partial\Omega} f, j = 1, 2. \tag{3.19}$$

Now we know that

$$\sigma_1(\Lambda_{q_j}) = \sqrt{g_2(x', 0, \xi')} = \gamma_j^{-\frac{1}{2}}\Big|_{\partial\Omega} \sigma_1(\Lambda_{\gamma_j})\, \gamma_j^{-\frac{1}{2}}\Big|_{\partial\Omega}$$

where $\sigma_m(A)$ denotes the principal symbol of a pseudodifferential operator of order m. So we deduce that

$$\gamma_1|_{\partial\Omega}\sqrt{g_2(x', 0, \xi')} = \sigma_1(\Lambda_{\gamma_1}) = \sigma_1(\Lambda_{\gamma_2}) = \gamma_2|_{\partial\Omega}\sqrt{g_2(x', 0, \xi')}$$

We then have that $\gamma_1|_{\partial\Omega} = \gamma_2|_{\partial\Omega}$. Therefore under the hypotheses of Theorem 3.3, from (3.19) we conclude that

$$\Lambda_{q_1} - \Lambda_{q_2} - \frac{1}{2}\left(\gamma_1^{-1}\frac{\partial\gamma_1}{\partial\nu}\right)\Big|_{\partial\Omega} - \frac{1}{2}\left(\gamma_2^{-1}\frac{\partial\gamma_2}{\partial\nu}\right)\Big|_{\partial\Omega} = 0.$$

Now we take the principal symbol of order zero in the above equation. By (12) we have that $\Lambda_{q_1} - \Lambda_{q_2}$ is a pseudodifferential operator of order zero and $\sigma_0(\Lambda_{q_1} - \Lambda_{q_2}) = 0$. We then obtain that

$$\frac{\partial\gamma_1}{\partial\nu}\Big|_{\partial\Omega} = \frac{\partial\gamma_2}{\partial\nu}\Big|_{\partial\Omega}.$$

Therefore we have that $\Lambda_{q_1} = \Lambda_{q_2}$ and by Theorem 2 $\partial^\alpha q_1|_{\partial\Omega} = \partial^\alpha q_2|_{\partial\Omega}\ \forall\ \alpha$. Since $q_j = \frac{\Delta\sqrt{\gamma_j}}{\sqrt{\gamma_j}}$ we arrive to the conclusion of the theorem. $\square$

In [S-U I] it was proved that knowing the principal symbol of Λ_γ $\sigma_1(\Lambda_\gamma) = \gamma|_{\partial\Omega}|\xi'|$ implies the stability estimate (2.24) for just continuous conductivities. The computation of $\sigma_0(\Lambda_\gamma - \gamma|_{\partial\Omega}\Lambda_1)$ where Λ_1 is the DN associated to the conductivity 1 leads to the following result in [S-U I].

Theorem 3.4. *Let γ_i be measurable functions such that*

$$0 < \frac{1}{\lambda} \leq \gamma_i \leq \lambda.$$

If γ_i are Lipschitz continuous in $\overline{\Omega}$ and for some β

$$\sup_{x\in\overline{\Omega}} |\nabla\gamma_i| \leq \beta$$

Then we have that the bounded linear map

$$B_i : H^{\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$$

where

$$B_i = \Lambda_{\gamma_i} - \gamma_i|_{\partial\Omega} \Lambda_1,$$

satisfies

$$\|B_1 - B_2\|_{\frac{1}{2},\frac{1}{2}} \le C(\lambda, \beta)\|\gamma_1 - \gamma_2\|_{W^{1,\infty}(\Omega)}$$

and

$$\|\gamma_1 - \gamma_2\|_{W^{1,\infty}(\partial\Omega)} \le C\|B_1 - B_2\|_{\frac{1}{2},\frac{1}{2}} + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}}$$

where $\| \quad \|_{\frac{1}{2},\frac{1}{2}}$ *and* $\| \quad \|_{\frac{1}{2},-\frac{1}{2}}$ *denotes the corresponding operator norms.*

This method has been generalized for several more complicated cases.

(a) (Anisotropic conductivities) Let $g(x) = g_{ij}(x)$ be a smooth Riemannian metric in $\overline{\Omega}$, i.e. g_{ij} is assumed to be a smooth, symmetric, positive definite matrix in $\overline{\Omega}$. Let Δ_g be the Laplace-Beltrami operator associated to g, i.e.

$$\Delta_g = \sum_{i,j=1}^{n} \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left(\sqrt{\det g} g^{ij} \frac{\partial}{\partial x_j} \right) \tag{3.20}$$

where $(g^{ij}) = (g_{ij})^{-1}$.

Let $f \in H^{\frac{1}{2}}(\partial\Omega)$. Let $u \in H^1(\Omega)$ be the unique solution of

$$\begin{aligned} -\Delta_g u &= 0 \\ u|_{\partial\Omega} &= f \end{aligned} \tag{3.21}$$

The DN map is defined by

$$\Lambda_g(f) = \sum_{i,j=1}^{n} g^{ij}\nu_j \left.\frac{\partial u}{\partial x_i}\right|_{\partial\Omega} \tag{3.22}$$

where ν_j denotes the components of ν. In section 1 of [L-U] the following result was proven: Let (x', x^n) denote boundary normal coordinates with respect to the metric g. Then the full symbol of Λ_g determines $\partial_{x_n}^{\alpha} g\big|_{\partial\Omega} \forall\ \alpha$.

We recall the definition of boundary normal coordinates. For each $r \in \partial\Omega$, let $\alpha_q : [0,\epsilon) \to \overline{\Omega}$ denote the limit-speed geodesic starting at r and normal to $\partial\Omega$. If $\{x^1,\dots,x^n\}$ are local coordinates for $\partial\Omega$ near $p \in \partial\Omega$, we can extend them smoothly to functions on a neighborhood of p in $\overline{\Omega}$ by letting them to be constant along each normal geodesic. If we define x^n to be the parameter along each α_r, then $\{x^1,\dots,x^n\}$ are coordinates called boundary normal coordinates.

The method of proof proceeds as before. Namely we find $B(x, D_{x'})$ a pseudodifferential operator of order 1 in x', depending smoothly on x_n such that we have the factorization

$$\begin{aligned} -\Delta_g &= D_{x^n}^2 + iE(x)D_{x^n} + Q(x, D_{x'}) \\ &= (D_{x^n} + iE(x) + iB(x, D_{x'}))(D_{x^n} - iB(x, D_{x'})) \end{aligned}$$

modulo smoothing.

Then we prove

$$\Lambda_g = B(x', 0, D_{x'}) \qquad \text{mod smoothing.}$$

Finally from the full symbol of $B(x', 0, D_{x'})$ we recover $\left.\frac{\partial^\alpha g}{(\partial x^n)^\alpha}\right|_{\partial\Omega}$. Notice that this statement depends on the coordinates (x', x^n).

(b) (The Schrödinger equation in the presence of a magnetic field.) Let $\vec{A} = (A_j(x),\dots,A_n(x))$, $A_j(x) \in C^\infty(\overline{\Omega})$ and $q \in C^\infty(\overline{\Omega})$ be real-valued functions. The Schrödinger equation in the presence of a magnetic field $\vec{A}$ and electrical potential q is given by

$$H_{\vec{A},q} = \sum_{j=1}^n \left(\frac{1}{i}\frac{\partial}{\partial x_j} + A_j\right)^2 + q(x). \tag{3.23}$$

We asume that 0 is not a Dirichlet eigenvalue of $H_{\vec{A},q}$. Let $u \in H^1(\Omega)$ be the unique solution of

$$\begin{aligned} H_{\vec{A},q} u &= 0 \qquad \text{in } \Omega \\ u|_{\partial\Omega} &= f \in H^{\frac{1}{2}}(\partial\Omega) \end{aligned} \tag{3.24}$$

Then the DN map is defined by

$$\Lambda_{\vec{A},q}(f) = \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} + i(\vec{A}\cdot\nu)f. \tag{3.23}$$

Sun observed in [Su I] that there is an obstruction to identifiability. Namely if we replace $\vec{A}$ by $\vec{A}+\nabla\varphi$ (a gauge transformation) with φ vanishing to first order on $\partial\Omega$, then

$$\Lambda_{\vec{A}+\nabla\varphi,q} = \Lambda_{\vec{A},q}.$$

This can be readily seen by observing that if u solves

$$H_{\vec{A},q}u = 0$$
$$u|_{\partial\Omega} = f$$

Then $e^{i\phi}u = v$ solves

$$H_{\vec{A}+\nabla\varphi,q}v = 0$$
$$v|_{\partial\Omega} = f$$

This shows that at best we can determine $\partial^\alpha(\text{ curl } \vec{A})\Big|_{\partial\Omega}$ $\forall\ \alpha$. In fact one of the results in [N-S-U] is

Theorem 3.5. *Under the conditions on $\vec{A}_i, q_i$, $i = 1, 2$ indicated above, if*

$$\Lambda_{\vec{A}_1,q_1} = \Lambda_{\vec{A}_2,q_2}$$

then there exists real-valued $\varphi \in C^\infty(\overline{\Omega})$ vanishing to first order on $\partial\Omega$ such that

$$\vec{A}_1 = \vec{A}_2 + \nabla\varphi$$

to infinite-order on $\partial\Omega$.

Proof. Introducing again coordinates (x', x_n) as in the case of $\Delta + q$ we write

$$H_{\vec{A},q} = D^2_{x_n} + iE(x)D_{x_n} + Q(x, D_{x'}) + \sum_{j=1}^{n} \widetilde{A}_j D_{x_j} + q$$

where $\widetilde{A}_j$ denotes the components of $\vec{A}$ in the coordinates (x', x_n). Now if we try a factorization as before we get

$$H_{\vec{A},q} = (D_{x_n} + iE(x) + \widetilde{A}_n + iB(x, D_{x'}))(D_{x_n} - iB(x, D_{x'}))$$

We first get rid of the term $\widetilde{A}_n$ by conjugating the operator $H_{\vec{A},q}$ with $e^{i\int_0^{x_n} \widetilde{A}_n(x',s)ds}$. Namely, we define

$$L_{\vec{A},q}u = e^{-i\int_0^{x_n} \widetilde{A}_n(x',s)ds} H_{\vec{A},q}(e^{i\int_0^{x_n} \widetilde{A}_n(x',s)ds}u).$$

Then

$$L_{\vec{A},q}u = D^2_{x_n} + iE(x)D_{x_n}u + Q(x, D_{x'})u$$
$$+ \sum_{j=1}^{n-1} C_j(x)D_{x_j}u + \widetilde{G}u$$

with C_j, $\widetilde{G}$ depending explicitly on $(\vec{A}, q)$. The point is that now a factorization of $L_{\vec{A},q}$ is indeed possible.

Proposition 1. *$\exists$ $B(x, D_{x'})$ a pseudodifferential operator of order 1 so that*

$$L_{\vec{A},q} = (D_{x_n} + iE(x) + iB(x, D_{x'}))(D_{x_n} - iB(x, D_{x'})) \qquad \textit{mod smoothing}$$

Following the same procedure as before one concludes that the DN map, $\mathcal{L}_{\vec{A},q}$, associated to $L_{\vec{A},q}$ determines $\partial^\alpha C_j|_{x_n=0}$, $\forall$ α, $j = 1, \ldots, n-1$. Now the result follows by relating $\mathcal{L}_{\vec{A},q}$ and $\Lambda_{\vec{A},q}$ and using the explicity form of the C_j's. Namely we have

$$\mathcal{L}_{\vec{A},q}(f) = \Lambda_{\vec{A},q}(f) - 2i\ \widetilde{A}_n\Big|_{x_n=0}.$$

B Layer stripping algorithm

Layer stripping algorithms have been developed for several inverse problems in one dimension (see for instance [Sym]) and in higher dimensions by several authors (see the Proceedings [C-K-N-S] and the references given there). For the Electrical Impedance Tomography the corresponding algorithm was developed in [S-C-I]. The idea is quite simple: We embed the domain $\partial\Omega$ in domains Ω_a $a \geq 0$ small with $\Omega_0 = \Omega$. In the case of the Schrödinger equation $-\Delta + q$, Ω_a is given by

$$\Omega_a = \{(x', x_n); x_n = a\}$$

so we have a family of DN maps $\Lambda_q^{(a)} = \Lambda_q|_{x_n=a}$. We know that we can determine $q\big|_{\partial\Omega_0}$ from Λ_q. Then we use the Riccati equation (3.8) (note that $B(x', a, D_{x'})$ is the DN map) to compute $\frac{d\Lambda_q}{dx_n}\Big|_{x_n=0}$. We now use the approximation

$$\Lambda_q|_{x_n=a} \approx \Lambda_q|_{x_n=0} + a\ \frac{d\Lambda_q}{dx_n}\Big|_{x_n=0}$$

In this way we can determine $\Lambda_q|_{x_n=a}$. We can then determine $q|_{x_n=a}$ and therefore we can use (3.9) again to write

$$\Lambda_q|_{x_n=a+\Delta a} \approx \Lambda_q|_{x_n=a} + \Delta a \left.\frac{d\Lambda_q}{dx_n}\right|_{x_n=a}.$$

Of course, the problem is that the Riccati equation is non-linear and there will be "blow-up" as x_n increases. Some regularization of this is needed. In [S-C-I] this was investigated for a ball $\mathbb{R}^2$ by dropping the high frequency modes. For a ball in $\mathbb{R}^2$ and a radial conductivity in the paper [Sy] the layer-stripping algorithm was regularized by adding a condition at the center of the ball. In this way in [Sy] a convergent layer stripping algorithm was obtained for radial conductivities.

We finish this section by showing another derivation of the Riccati equation (3.8) is satisfied exactly, not just up to a smoothing operator as shown earlier. We follow [S-C-I].

We solve the family of Dirichlet problems

(3.26) $$L_q u = (D_{x_n}^2 + iE(x)D_{x_n} + Q + q)u(x,a) = 0 \text{ in } \Omega_a$$

(3.27) $$u(x,a)|_{x_n=a} = f(x')$$

We define the family of DN maps

(3.28) $$\Lambda_q^{(a)}(f) = -\left.\frac{\partial u}{\partial x_n}\right|_{x_n=a}.$$

We differentiate (3.26) and (3.27) with respect to a to obtain

$$L_q \frac{\partial u}{\partial a} = 0$$
$$\left.\frac{\partial u}{\partial a}\right|_{x_n=a} = \left.\frac{\partial u}{\partial x_n}\right|_{x_n=a} = -\Lambda_q^{(a)}(f)$$

Therefore

(3.29) $$(\Lambda_q^{(a)})^2(f) = \left.\frac{\partial^2 u}{\partial x_n \partial a}\right|_{x_n=a}$$

Now we differentiate (3.28) with respect to a. We get

3.30 $$\left(\frac{d\Lambda_q^{(a)}}{da}\right)(f) = \frac{d}{da}(\Lambda_q^{(a)} f)$$

$$
\begin{aligned}
&= \frac{d}{da}\left(- \left.\frac{\partial u}{\partial x_n}\right|_{x_n=a} \right) \\
&= -\left(\left(\frac{\partial}{\partial x_n} + \frac{\partial}{\partial a} \right) \left(\frac{\partial}{\partial x_n} \right) \right) (u)|_{x_n=a} \\
&= -iE(x)\Lambda_q^a f + Qf + qf - (\Lambda_q^{(a)})^2 f
\end{aligned}
$$

by using (3.26), (3.27), (3.28) and (3.29). So we get

$$
\frac{d\Lambda_q^{(a)}}{da} + E\Lambda_q^a + (\Lambda_q^{(a)})^2 - Q - q = 0 \tag{3.31}
$$

which is equation(3.8) since $i[D_{x_n}, B(x, D'_x)]f(x') = \partial_{x_n} B(x, D_{x'})f(x')$.

§4. Exponentially Growing Solutions

If we look for "plane wave" exponential solutions to Laplace's equation, i.e., if we seek

$$u = e^{x\cdot\zeta} \qquad \zeta \in \mathbf{C}^n \tag{4.1}$$

which satisfy

$$\Delta u = 0,$$

then we must necessarily have

$$\zeta \cdot \zeta = 0\ ; \tag{4.2}$$

conversely (4.1) solves Laplace's equation whenever (4.2) is satisfied. As any nontrivial solutions to (4.2) will have non-zero real part, the corresponding solution (4.1) will grow exponentially in most directions. The search for solutions analogous to (4.1) with the Laplacian replaced by $\Delta + q$ will be the main subject of this section. The utility of exponentially growing solutions in solving the inverse conductivity problem was first observed by Calderón in [C]. We begin by exhibiting Calderón's proof of injectivity of the linearized inverse boundary value problem.

We recall that the mapping Λ defined by

$$\gamma \overset{\Lambda}{\mapsto} \Lambda_\gamma$$

is an analytic map from $L^\infty(\Omega)$ to $\mathcal{BL}_{1/2,-1/2}$, the vector space of bounded linear maps from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$ endowed with the operator norm. We denote by $D\Lambda_\gamma[\delta\gamma]$ the Frechet derivative of Λ at γ acting on the perturbation $\delta\gamma$. Calderón proved the following result.

Theorem 4.1. *The Frechet derivative of Λ at $\gamma = 1$, $D\Lambda_1[\ \cdot\ ]$, is injective. That is, if*

$$D\Lambda_1[\delta\gamma] = 0 \textit{ for some } \delta\gamma \in L^\infty(\Omega)$$

then

$$\delta\gamma = 0$$

Proof Let $\gamma = \gamma(t)$ be a smooth curve in $L^\infty(\Omega)$ and let $u(t)$ and $v(t)$ satisfy, for each t

$$\begin{array}{cc} L_\gamma u = 0, & L_\gamma v = 0 \\ u|_{\partial\Omega} = f, & v|_{\partial\Omega} = g \\ \gamma\frac{\partial u}{\partial \nu}|_{\partial\Omega} = \alpha, & \gamma\frac{\partial v}{\partial \nu}|_{\partial\Omega} = \beta, \end{array} \tag{4.3}$$

then integration by parts gives the identity

$$\int_{\partial\Omega} (f\beta(t) - g\alpha(0))\ dS(x) = \int_\Omega (\nabla u(0)^T\gamma(t)\nabla v(t) - \nabla u(0)^T\gamma(0)\nabla v(t))\ dx.$$

Differentiation with respect to t at $t = 0$ gives

$$\int_{\partial\Omega} f\,\dot{\beta}(0)\ dS(x) = \int_\Omega \nabla u(0)^T\,\dot{\gamma}(0)\nabla v(0)\ dx \quad ,$$

where $\bullet$ denotes $\frac{d}{dt}$ and u and v satisfy (4.3). Since $\dot{\beta}(0)$ equals $D\Lambda_{\gamma(0)}[\dot{\gamma}(0)]g$ this identity may also be written

$$\langle f, D\Lambda_{\gamma(0)}[\dot{\gamma}(0)]g\rangle = \int_\Omega \nabla u(0)^T\,\dot{\gamma}(0)\nabla v(0)\ dx \quad ,$$

for every f and $g \in H^{1/2}(\partial\Omega)$. By taking $\dot{\gamma}(0) = \delta\gamma$ it now follows that he equation

$$D\Lambda_{\gamma(0)}[\delta\gamma] = 0$$

is equivalent to

$$\int_\Omega \nabla u^T\delta\gamma\nabla v\ dx = 0 \tag{4.4}$$

for every u and $v \in H^1(\Omega)$ which satisfy $L_{\gamma(0)}u = L_{\gamma(0)}v = 0$. If we further restrict to $\gamma(0) = 1$ then (4.4) must hold for every pair of harmonic functions u and v. A very natural set of choices for u and v are those exponentials (4.1) which satisfy (4.2), i.e., let

$$u = e^{x\cdot\zeta_1}, \qquad v = e^{x\cdot\zeta_2}$$

with $\zeta_j \cdot \zeta_j = 0$. Then it follows from (4.4) that

$$\zeta_1\cdot\zeta_2\int_\Omega e^{x\cdot(\zeta_1+\zeta_2)}\delta\gamma\ dx = 0$$

or

$$\frac{(\zeta_1+\zeta_2)\cdot(\zeta_1+\zeta_2)-(\zeta_1-\zeta_2)\cdot(\zeta_1-\zeta_2)}{4}\int_\Omega e^{x\cdot(\zeta_1+\zeta_2)}\delta\gamma\ dx=0\ .$$

We now require that

$$\zeta_1+\zeta_2=ik,\quad k\in\mathbf{R}^n,$$

and note that since $\zeta_1\cdot\zeta_1=\zeta_2\cdot\zeta_2=0$

$$-(\zeta_1-\zeta_2)\cdot(\zeta_1-\zeta_2)=(\zeta_1+\zeta_2)\cdot(\zeta_1+\zeta_2)=-k\cdot k\quad .$$

Hence $D\Lambda_1[\delta\gamma]=0$ implies that

$$k\cdot k\int_\Omega e^{ix\cdot k}\delta\gamma\ dx=0\quad ,$$

which again implies that

$$\text{supp}\ (\widehat{\chi_\Omega\delta\gamma})\subset\{0\}\quad ,$$

with χ_Ω denoting the characteristic function of the set Ω. However, $\chi_\Omega\delta\gamma$ is an element of $L^2(\mathbf{R}^n)$, so that $\widehat{\chi_\Omega\delta\gamma}$ is in $L^2(\mathbf{R}^n)$ and therefore cannot be supported at a single point. As a consequence

$$\chi_\Omega\delta\gamma=0\ ,$$

which proves that $D\Lambda_1[\ \cdot\ ]$ is injective. $\square$

Before proceeding, we formulate the analog of Theorem 4.1 for Schrödinger operators. Let $\mathcal{C}_q$ denote the Cauchy data for $\Delta+q$, defined by

$$\mathcal{C}_q=\{(f,g)\in H^{1/2}(\partial\Omega)\times H^{-1/2}(\partial\Omega):\\ \exists v\in H^1(\Omega)\text{ with }\Delta v+qv=0\text{ in }\Omega,\text{ and }v|_{\partial\Omega}=f,\ \frac{\partial v}{\partial\nu}|_{\partial\Omega}=g\}\ .$$

The map

$$L^\infty(\Omega)\ni q\overset{\mathcal{C}}{\mapsto}\mathcal{C}_q\in\{\text{ linear subspaces of }H^{\frac12}(\partial\Omega)\times H^{-\frac12}(\partial\Omega)\}$$

is also real analytic, and we denote by $D\mathcal{C}_q$, its Frechet derivative at q.

Theorem 4.2. *The Frechet derivative of $\mathcal{C}$ at $q = 0$, $D\mathcal{C}_0[\,\cdot\,]$, is injective.*

Remark. For any fixed Ω and q sufficiently small ($\|q\|_{L^\infty}$ < smallest eigenvalue of $-\Delta$ with Dirichlet boundary conditions) $\mathcal{C}_q$ is the graph of the Dirichlet to Neumann map, Λ_q, corresponding to the operator $\Delta + q$. Hence, the Frechet derivative acting on the perturbation δq is given by

$$D\mathcal{C}_q[\delta q] = \{(0, D\Lambda_q[\delta q]f) : \ f \in H^{1/2}(\partial\Omega)\} \ ,$$

and the statement that $D\mathcal{C}_0[\,\cdot\,]$ is injective is equivalent to the statement $D\Lambda_q[\,\cdot\,]$ is injective at $q = 0$. □

The approach that we will use to prove identifiability later in § 5 is based on exponential solutions which are constructed in a way that naturally extends the previous construction for the Laplacian. To construct these solutions we shall make use of the following norms, defined for any $u \in C_0^\infty(\mathbf{R}^n)$ and any $-\infty < \delta < \infty$:

$$\|u\|_{L^2_\delta} = \left(\int_{\mathbf{R}^n} \left(1 + |x|^2\right)^\delta |u|^2 \, dx\right)^{1/2} .$$

The space L^2_δ is defined as the completion of $C_0^\infty(\mathbf{R}^n)$ with respect to the norm $\|\cdot\|_{L^2_\delta}$. The main theorem in this section is:

Theorem 4.3. *Let $-1 < \delta < 0$. There exists $\epsilon = \epsilon(\delta)$ and $C = C(\delta)$ such that, for every $q \in L^2_{\delta+1} \cap L^\infty$ and every $\zeta \in \mathbf{C}^n$ satisfying*

$$\zeta \cdot \zeta = 0 \quad \textit{and} \tag{4.5}$$

$$\frac{\|(1 + |x|^2)^{1/2} q\|_{L^\infty} + 1}{|\zeta|} \le \varepsilon \ , \tag{4.6}$$

there exists a unique solution to

$$\Delta u + qu = 0 \qquad \textit{in } \mathbf{R}^n \tag{4.7}$$

of the form

$$u = e^{x\cdot\zeta}\bigl(1 + \psi(x, \zeta)\bigr) \tag{4.8}$$

with $\psi(x,\zeta) \in L^2_\delta$. *Furthermore,*

$$\|\psi\|_{L^2_\delta} \le \frac{C}{|\zeta|}\|q\|_{L^2_{\delta+1}}. \tag{4.9}$$

This theorem has a counterpart for the conductivity problem, which is obtained by envoking the correspondence in Theorem 0.6 between the Schrödinger equation and the conductivity equation. The statement is

Theorem 4.4. *Let* $-1 < \delta < 0$. *There exists* $\epsilon = \epsilon(\delta)$ *and* $C = C(\delta)$ *such that, for every positive* γ *with* $\frac{\Delta\gamma^{1/2}}{\gamma^{1/2}} \in L^2_{\delta+1} \cap L^\infty$ *and every* $\zeta \in \mathbf{C}^n$ *satisfying*

$$\zeta \cdot \zeta = 0 \quad \textit{and}$$

$$\frac{\|(1+|x|^2)^{1/2}\frac{\Delta\gamma^{1/2}}{\gamma^{1/2}}\|_{L^\infty} + 1}{|\zeta|} \le \epsilon\ ,$$

there exists a unique solution to

$$L_\gamma u = 0$$

of the form

$$u = \gamma^{-1/2} e^{x\cdot\zeta}\big(1 + \psi(x,\zeta)\big)$$

with $\psi(x,\zeta) \in L^2_\delta$. *Furthermore,*

$$\|\psi(x,\zeta)\|_{L^2_\delta} \le \frac{C}{|\zeta|}\Big\|\frac{\Delta\gamma^{1/2}}{\gamma^{1/2}}\Big\|_{L^2_{\delta+1}}.$$

Most of the work necessary for the proof of Theorem 4.3 is associated with establishing the following proposition.

Proposition 4.1. *Suppose that* $\zeta\cdot\zeta = 0$, $|\zeta| \ge c > 0$, $f \in L^2_{\delta+1}$ *and* $-1 < \delta < 0$. *There exists a unique* $\varphi \in L^2_\delta$ *such that*

$$(\Delta + 2\zeta\cdot\nabla)\varphi = f. \tag{4.10}$$

Moreover,

$$\|\varphi\|_{L^2_\delta} \le \frac{C(\delta,c)}{|\zeta|}\|f\|_{L^2_{\delta+1}}. \tag{4.11}$$

We postpone the proof of this proposition to the end of this chapter, instead we first show how it may be applied for the

Proof of Theorem 4.3

We seek u of the form

$$u = e^{x\cdot\zeta}(1+\psi)$$

satisfying

$$(\Delta + q)\{e^{x\cdot\zeta}(1+\psi)\} = 0$$

or

$$\Delta\psi + 2\zeta\cdot\nabla\psi = -q - q\psi. \tag{4.12}$$

To solve (4.12), we define

$$\psi_{-1} = 1$$

and we recursively define ψ_j by

$$(\Delta + 2\zeta\cdot\nabla)\psi_j = -q\psi_{j-1} \quad \text{for } j \geq 0. \tag{4.13}$$

We claim that

$$\psi := \sum_{j=0}^{\infty}\psi_j \tag{4.14}$$

is the desired solution. It needs to be proved that the functions ψ_j, $j \geq 0$, are well defined, and that the series (4.14) is convergent. We may without loss of generality restrict our attention to $\epsilon < 1$, so that we only consider ζ for which $|\zeta| \geq 1$. Since $q \in L^2_{\delta+1}$ and $\psi_{-1} = 1$ it follows from Proposition 4.1 that there exists a unique $\psi_0 \in L^2_\delta$ that solves (4.13) with $j = 0$. This ψ_0 furthermore satisfies

$$\|\psi_0\|_{L^2_\delta} \leq \frac{C(\delta)}{|\zeta|}\|q\|_{L^2_{\delta+1}} \ . \tag{4.15}$$

If v is an element in L^2_δ, then the fact that $(1+|x|^2)^{1/2}q$ is in L^∞ immediately implies that qv is in $L^2_{\delta+1}$ with the estimate

$$\|qv\|_{L^2_{\delta+1}} \leq \|(1+|x|^2)^{1/2}q\|_{L^\infty}\|v\|_{L^2_\delta} \ .$$

Using this observation in connection with Proposition 4.1 we conclude that if ψ_{j-1} is in L^2_δ then there exists a unique solution to (4.13) in L^2_δ. This solution ψ_j furthermore satisfies

$$\begin{aligned}\|\psi_j\|_{L^2_\delta} &\le \frac{C(\delta)}{|\zeta|}\|q\psi_{j-1}\|_{L^2_{\delta+1}} \\ &\le \left(\frac{C(\delta)\|(1+|x|^2)^{1/2}q\|_{L^\infty}}{|\zeta|}\right)\|\psi_{j-1}\|_{L^2_\delta} \quad .\end{aligned} \tag{4.16}$$

An induction argument based on the estimates (4.15) and (4.16) now gives that ψ_j, $j \ge 0$, are all elements of L^2_δ and satify the estimates

$$\|\psi_j\|_{L^2_\delta} \le \frac{C(\delta)}{|\zeta|}\theta^j\|q\|_{L^2_{\delta+1}} \quad \text{with} \quad \theta = \frac{C(\delta)\|(1+|x|^2)^{1/2}q\|_{L^\infty}}{|\zeta|} \quad .$$

By selecting ϵ sufficiently small, say that $\theta < 1/2$, we now obtain that the series (4.14) is convergent, with the bound

$$\|\psi\|_{L^2_\delta} \le 2\frac{C(\delta)}{|\zeta|}\|q\|_{L^2_{\delta+1}} \quad .$$

This completes the proof of the existence part of Theorem 4.3

To verify the uniqueness of the solution ψ (and therefore of u), suppose that

$$\Delta\psi + 2\zeta\cdot\nabla\psi = -q - q\psi$$

and

$$\Delta\tilde{\psi} + 2\zeta\cdot\nabla\tilde{\psi} = -q - q\tilde{\psi} \ ,$$

with ψ and $\tilde{\psi} \in L^2_\delta$. Then

$$\Delta(\tilde{\psi}-\psi) + 2\zeta\cdot\nabla(\tilde{\psi}-\psi) = q(\psi-\tilde{\psi}) \ ,$$

so that according to Proposition 4.1

$$\begin{aligned}\|\tilde{\psi}-\psi\|_{L^2_\delta} &\le \frac{C\|(1+|x|^2)^{1/2}q\|_{L^\infty}}{|\zeta|}\|\tilde{\psi}-\psi\|_{L^2_\delta} \\ &\le \frac{1}{2}\|\tilde{\psi}-\psi\|_{L^2_\delta} \ ,\end{aligned}$$

which can only occur if

$$\|\tilde{\psi}-\psi\|_{L^2_\delta} = 0 \quad .$$

□

It is not exactly Theorem 4.3 we use later on in our proof of identifiability, rather it is the following version for a bounded domain.

Corollary 4.3. *Let Ω be a bounded domain in $\mathbf{R}^n$. There exist constants ϵ and C such that for every $q \in L^\infty(\Omega)$ and every $\zeta \in \mathbf{C}^n$ satisfying*

$$\zeta \cdot \zeta = 0 \quad \textit{and}$$

$$\frac{\|q\|_{L^\infty} + 1}{|\zeta|} \leq \epsilon \ ,$$

there exists a solution $u \in H^1(\Omega)$ to

$$\Delta u + qu = 0 \ \ \textit{in } \Omega$$

of the form

$$u = e^{x \cdot \zeta}\big(1 + \psi(x, \zeta)\big)$$

with

$$\|\psi\|_{L^2(\Omega)} \leq \frac{C}{|\zeta|}\|q\|_{L^2(\Omega)} \ \ \textit{and}$$
$$\|\psi\|_{H^1(\Omega)} \leq C\|q\|_{L^2(\Omega)} \ \ .$$

Proof

Define

$$\tilde{q} = \begin{cases} \mathrm{q} & \text{in } \Omega \\ 0 & \text{in } \mathbf{R}^n \setminus \Omega \end{cases} \ \ .$$

We may apply Theorem 4.3 to $\tilde{q}$, say with $\delta = 1/2$. This way we obtain the existence of a solution to $\Delta u + \tilde{q}u = 0$ in $\mathbf{R}^n$ (and therefore a solution to $\Delta u + qu = 0$ in Ω) of the form $u = e^{x \cdot \zeta}(1 + \psi(x, \zeta))$ with

$$\|\psi\|_{L^2(\Omega)} \leq \frac{C}{|\zeta|}\|q\|_{L^2(\Omega)} \ . \tag{4.17}$$

By interior elliptic regularity estimates it follows that $u \in H^1(\Omega)$. It only remains to prove the estimate concerning the H^1 norm of ψ. As a means to obtain this estimate we establish a particular interior estimate for solutions to

$$\Delta v = -F \quad \text{in } \mathbf{R}^n \ \ ,$$

namely

$$\|v\|_{H^1(\Omega)} \leq C\big(\|F\|_{H^{-1}(\Omega')} + \|v\|_{L^2(\Omega')}\big) \ \ , \tag{4.18}$$

provided $\Omega \subset\subset \Omega'$. Let $\chi \in C_0^1(\mathbf{R}^n)$, $0 \le \chi \le 1$ be such that $\chi \equiv 1$ on Ω and $\operatorname{supp} \chi \subset \Omega'$, then integration by parts and use of Hölder's inequality yields

$$\begin{aligned} \int_{\mathbf{R}^n} \chi^2 |\nabla v|^2 \, dx &= \int_{\mathbf{R}^n} F \chi^2 v \, dx + 2 \int_{\mathbf{R}^n} \chi \nabla \chi \cdot \nabla v \; v \, dx \\ &\le C \|F\|^2_{H^{-1}(\Omega')} + \frac{1}{8} \|\chi^2 v\|^2_{H^1(\Omega')} \\ &\quad + \frac{1}{4} \int_{\mathbf{R}^n} \chi^2 |\nabla v|^2 \, dx + C \int_{\mathbf{R}^n} v^2 |\nabla \chi|^2 \, dx \quad . \end{aligned} \tag{4.19}$$

On the other hand

$$\begin{aligned} \|\chi^2 v\|^2_{H^1(\Omega')} &= \int_{\mathbf{R}^n} |\nabla(\chi^2 v)|^2 \, dx + \int_{\mathbf{R}^n} |\chi^2 v|^2 \, dx \\ &\le 2 \int_{\mathbf{R}^n} \chi^4 |\nabla v|^2 \, dx + 2 \int_{\mathbf{R}^n} v^2 |\nabla \chi^2|^2 \, dx \\ &\quad + \int_{\mathbf{R}^n} |\chi^2 v|^2 \, dx \\ &\le 2 \int_{\mathbf{R}^n} \chi^2 |\nabla v|^2 \, dx + C \|v\|^2_{L^2(\Omega')} \quad . \end{aligned} \tag{4.20}$$

A combination of (4.19) and (4.20) gives

$$\begin{aligned} \int_{\mathbf{R}^n} \chi^2 |\nabla v|^2 \, dx &\le C \big(\|F\|^2_{H^{-1}(\Omega')} + \|v\|^2_{L^2(\Omega')} \big) \\ &\quad + \frac{1}{2} \int_{\mathbf{R}^n} \chi^2 |\nabla v|^2 \, dx \quad , \end{aligned}$$

and therefore

$$\begin{aligned} \frac{1}{2} \int_{\Omega} |\nabla v|^2 \, dx &\le \frac{1}{2} \int_{\mathbf{R}^n} \chi^2 |\nabla v|^2 \, dx \\ &\le C \big(\|F\|^2_{H^{-1}(\Omega')} + \|v\|^2_{L^2(\Omega')} \big) \quad . \end{aligned}$$

This immediately leads to the estimate (4.18).

Going back to the equation (4.12) we get that

$$\Delta \psi = -2\zeta \cdot \nabla \psi - \tilde{q} - \tilde{q} \psi \quad \text{in } \mathbf{R}^n \ ,$$

and the estimate (4.18) thus gives

$$\|\psi\|_{H^1(\Omega)} \le C \left(\|2\zeta \cdot \nabla \psi + \tilde{q} + \tilde{q}\psi\|_{H^{-1}(\Omega')} + \|\psi\|_{L^2(\Omega')} \right) \ , \tag{4.21}$$

with $\Omega \subset\subset \Omega'$. On the other hand we also have

$$\begin{aligned}\|2\zeta \cdot \nabla\psi + \tilde{q} + \tilde{q}\psi\|_{H^{-1}(\Omega')} &\leq 2|\zeta|\|\psi\|_{L^2(\Omega')} + \|\tilde{q}\|_{L^2(\Omega')} + \|\tilde{q}\|_{L^2(\Omega')}\|\psi\|_{L^2(\Omega')} \\ &= 2|\zeta|\|\psi\|_{L^2(\Omega')} + \|q\|_{L^2(\Omega)} + \|q\|_{L^2(\Omega)}\|\psi\|_{L^2(\Omega')} ,\end{aligned} \tag{4.22}$$

and

$$\|\psi\|_{L^2(\Omega')} \leq \frac{C}{|\zeta|}\|\tilde{q}\|_{L^2(\Omega')} = \frac{C}{|\zeta|}\|q\|_{L^2(\Omega)} . \tag{4.23}$$

The estimate (4.23) is obtained by replacing Ω by Ω' in the estimate (4.17) (the constant C changes). A combination of (4.21)-(4.23) yields

$$\|\psi\|_{H^1(\Omega)} \leq C\left(\|q\|_{L^2(\Omega)} + \frac{\|q\|^2_{L^2(\Omega)}}{|\zeta|} + \frac{\|q\|_{L^2(\Omega)}}{|\zeta|}\right) ,$$

and since the assumption about $|\zeta|$ implies that $1/|\zeta| \leq C\min(1, \|q\|^{-1}_{L^2(\Omega)})$, we immediately get

$$\|\psi\|_{H^1(\Omega)} \leq C\|q\|_{L^2(\Omega)} ,$$

as desired. □

We now return to the

Proof of Proposition 4.1

We first prove uniqueness. Suppose that $w \in L^2_\delta$ and

$$\Delta w + 2\zeta \cdot \nabla w = 0.$$

Fourier transformation gives

$$(-|\xi|^2 + 2\zeta \cdot i\xi)\hat{w} = 0. \tag{4.24}$$

As this equation is invariant under rotations, we may without loss of generality assume that

$$\zeta = s\begin{pmatrix}1\\0\\0\\.\\.\\.\end{pmatrix} + is\begin{pmatrix}0\\1\\0\\.\\.\\.\end{pmatrix} = se_1 + ise_2, \quad s = \frac{|\zeta|}{\sqrt{2}} ,$$

in which case (4.24) is equivalent to

$$\left[\left(\xi_1^2 + (\xi_2 - s)^2 + \xi_3^2 \ldots + \xi_n^2 - s^2\right) + 2is\xi_1\right] \cdot \hat{w} = 0. \tag{4.25}$$

The content of (4.25) is that the tempered distribution $\hat{w}$ is supported on the manifold $\mathcal{M}(s)$, where $\mathcal{M}(s)$ denotes the codimension 2 sphere which arises as the intersection of the plane $\xi_1 = 0$ and the n-1 dimensional sphere with center se_2 and radius s. Whenever misunderstandings are excluded we shall for brevity use the notation $\mathcal{M}$ in place of $\mathcal{M}(s)$.

We will apply Plancherel's theorem to $\hat{w}$, but, in order to do so, we first smooth the distribution $\hat{w}$ by introducing

$$\hat{w}_\varepsilon(\cdot) = \varepsilon^{-n}\widehat{\beta(|x|)}\left(\frac{\cdot}{\epsilon}\right) * \hat{w}(\cdot)$$

where

$$\beta(|x|) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbf{R}^n} \chi(|\xi|^2)e^{-i\xi x}\, d\xi \quad \text{with}$$
$$\chi \in C^\infty([0,\infty)) \ \text{ and } \ \text{supp } \chi \subset [0,1] \ .$$

From these definitions it follows immediately that

$$\widehat{\beta(|x|)}(\xi) = \chi(|\xi|^2) \ .$$

We furthermore normalize β by the requirement that

$$\int_{\mathbf{R}^n} \widehat{\beta(|x|)}(\xi)\, d\xi = \int_{\mathbf{R}^n} \chi(|\xi|^2)\, d\xi = 1 \ .$$

It straightforward to see that

$$\lim_{\epsilon\downarrow 0} \hat{w}_\epsilon = \hat{w} \ ,$$
$$\text{supp } \hat{w}_\varepsilon \subset N_\varepsilon\left(\mathcal{M}(s)\right) = \{\xi |\ \text{dist}(\xi, \mathcal{M}(s)) \le \varepsilon\} \ ,$$
$$\left(\varepsilon^{-n}\widehat{\beta(|\cdot|)}\left(\frac{\xi}{\epsilon}\right)\right)^\vee (x) = \beta(\epsilon|x|) \ .$$

For any $\varphi \in \mathcal{S}(\mathbf{R}^n)$

$$\begin{aligned}\langle w, \varphi\rangle &= \langle \hat{w}, \check{\varphi}\rangle \\ &= \lim_{\epsilon\downarrow 0} \int_{\mathbf{R}^n} \hat{w}_\epsilon \check{\varphi}\, dx\end{aligned}$$

so that

$$|\langle w,\varphi\rangle| \le \overline{\lim_{\epsilon\downarrow 0}}\ \epsilon \left(\int_{N_\varepsilon} |\hat{w}_\epsilon|^2 d\xi\right)^{1/2} \cdot \left(\frac{1}{\varepsilon^2}\int_{N_\epsilon} |\check{\varphi}(\xi)|^2 d\xi\right)^{1/2}$$

As $\check{\varphi}$ is smooth and the (volume of N_ϵ) $/\epsilon^2$ converges to a constant times the surface area of $\mathcal{M}$,

$$|\langle w,\varphi\rangle| \le C\left(\overline{\lim_{\epsilon\downarrow 0}}\ \epsilon \|\hat{w}_\epsilon\|_{L^2}\right)\ \left(\int_{\mathcal{M}(s)} |\check{\varphi}(\xi)|^2 dS(\xi)\right)^{1/2}. \tag{4.26}$$

Moreover,

$$\begin{aligned}\left(\frac{1}{2\pi}\right)^n \|\hat{w}_\epsilon\|^2_{L^2} &= \|w_\epsilon\|^2_{L^2}\\ &= \int_{\mathbf{R}^n} |\beta(\epsilon|x|)|^2\ |w(x)|^2\ dx\\ &\le \sup(|\beta^2(\epsilon|x|)(1+|x|^2)^{-\delta}|)\cdot \|w\|^2_{L^2_\delta}.\end{aligned}$$

As $\beta\in\mathcal{S}(\mathbf{R}^n)$, and $\delta<0$, it therefore follows that

$$\begin{aligned}\|\hat{w}_\epsilon\|^2_{L^2} &\le C\sup(|(1+\epsilon^2|x|^2)^\delta|\cdot|(1+|x|^2)^{-\delta}|)\cdot\|w\|^2_{L^2_\delta}\\ &\le C\epsilon^{2\delta}\|w\|^2_{L^2_\delta}.\end{aligned}$$

Returning to (4.26)

$$|\langle w,\varphi\rangle| \le C\ \overline{\lim_{\epsilon\downarrow 0}}(\epsilon\cdot\epsilon^\delta)\|w\|_{L^2_\delta}\left(\int_{\mathcal{M}(s)} |\check{\varphi}(\xi)|^2 dS(\xi)\right)^{1/2}.$$

Since $\delta>-1$, it therefore follows that

$$\langle w,\varphi\rangle = 0$$

for every $\varphi\in\mathcal{S}$, so that $w=0$.

We turn to prove existence of a solution to (4.10). Suppose for now that $f\in\mathcal{S}(\mathbf{R}^n)$ and define

$$\hat{w}(\xi) = \frac{\hat{f}(\xi)}{-|\xi|^2+2i\zeta\cdot\xi}.$$

We shall prove that w is well defined and satisfies the estimate

$$\|w\|_{L^2_\delta} \le \frac{C}{|\zeta|}\|f\|_{L^2_{\delta+1}}. \tag{4.27}$$

Once this estimate is established we can dispense with the assumption that $f \in \mathcal{S}(\mathbf{R}^n)$ by continuity. As we did in the uniqueness proof, we may assume that

$$\zeta = s(e_1 + ie_2) \ , \quad s = \frac{|\zeta|}{\sqrt{2}} \ ,$$

and therefore

$$-|\xi|^2 + 2i\zeta \cdot \xi = \xi_1^2 + (\xi_2 - s)^2 + \xi_3^2 \ldots + \xi_n^2 - s^2 + 2is\xi_1 = P(\xi, s).$$

With this definition of the polynomial P it is easy to see that

$$P(\xi, s) = s^2 P(\xi/s, 1) \ .$$

As before we denote

$$N_r(\mathcal{M}(s)) = \{\xi \in \mathbf{R}^n | \text{ dist } (\xi, \mathcal{M}(s)) \leq r\}$$

and we define an open cover of $\mathbf{R}^n$ by

$$\begin{aligned}
\mathcal{O}_1(s) &= \mathbf{R}^n \backslash N_{s/2n}(\mathcal{M}(s)) \\
\mathcal{O}_2(s) &= \{|\xi_2 - s| \geq s/2n\} \cap \overset{\circ}{N}_s(\mathcal{M}(s)) \\
\mathcal{O}_j(s) &= \{|\xi_j| \geq s/2n\} \cap \overset{\circ}{N}_s(\mathcal{M}(s)) \text{ for } j > 2.
\end{aligned}$$

It is useful to note that $\mathcal{M}(s) = s\mathcal{M}(1)$ and that $\mathcal{O}_j(s) = s\mathcal{O}_j(1)$.
Let $\chi_j(\xi)$ be a partition of unity subordinate to this open cover, so that

$$\hat{w}(\xi) = \sum_{j=1}^{n} \hat{w}_j(\xi) = \sum_{j=1}^{n} \frac{\chi_j(\xi)\hat{f}(\xi)}{P(\xi, s)} \tag{4.28}$$

Since $\mathcal{O}_1(1)$ is disjoint from $\mathcal{M}(1)$ and since $P(\xi, 1) \to \infty$ as $|\xi| \to \infty$ there exists a constant c such that

$$P(\xi, 1) \geq c > 0 \quad \forall \xi \in \mathcal{O}_1(1) \ .$$

For $\xi \in \mathcal{O}_1(s)$ this leads to the estimate

$$P(\xi, s) = s^2 P(\xi/s, 1) \geq cs^2$$

so that

$$\|w_1\|_{L^2_\delta} \le \|w_1\|_{L^2} \le \frac{1}{cs^2}\|f\|_{L^2} \le \frac{1}{cs^2}\|f\|_{L^2_{\delta+1}} \quad . \tag{4.29}$$

Here we are use the facts that $\delta < 0$ and $\delta + 1 > 0$. Since our hypothesis guarantees that $|\zeta| = \sqrt{2}s$ is greater than some $c > 0$, (4.29) gives the desired estimate for w_1.

To estimate each of the $w_j, j = 2, ..., n$ we first introduce new coordinates in $\mathcal{O}_j(s)$ by

$$\eta_1 = 2\xi_1$$

$$\eta_\ell = \xi_\ell \text{ for } \ell \ne 1,\ j$$

$$\eta_j = \frac{\xi_1^2 + (\xi_2 - s)^2 + \xi_3^2 \ldots + \xi_n^2 - s^2}{s}$$

In terms of these new coordinates

$$\hat{w}_j(\eta) = \frac{\chi_j(\xi)\hat{f}(\xi)}{s(\eta_j + i\eta_1)} \quad .$$

The Jacobian of this coordinate transformation on $\mathcal{O}_j(s)$ is easily calculated to be

$$\text{Det}\left(\frac{\partial\eta}{\partial\xi}\right) = \frac{4\xi_j}{s} \quad \text{for } j \ne 2 \quad ,$$

and

$$\text{Det}\left(\frac{\partial\eta}{\partial\xi}\right) = \frac{4(\xi_2 - s)}{s} \quad \text{for } j = 2 \quad .$$

These expressions are in all cases bounded from above and below on $\mathcal{O}_j(s)$, $j = 2, ..., n$ independently of s. At this point we shall make use of the following three results, the proofs of which will be given later.

Lemma 4.1. *The maps Z_j defined by*

$$(Z_j f)(\xi) = \left(\frac{\hat{f}}{\xi_j + i\xi_1}\right)^\vee \qquad f \in \mathcal{S}(\mathbf{R}^n)$$

are bounded from $L^2_{\delta+1}$ to L^2_δ.

Lemma 4.2. *For any $\chi \in C_0^\infty(\mathbf{R}^n)$ and any $f \in \mathcal{S}(\mathbf{R}^n)$*

$$\|\big(\chi(\xi)\hat{f}(\xi)\big)^\vee\|_{L^2_{\delta+1}} \le C\|f\|_{L^2_{\delta+1}} \quad ,$$

where the constant C depends on χ, but is independent of f.

Lemma 4.3. *Let $\mathcal{O}$ and $\mathcal{O}'$ be open subsets of $\mathbf{R}^n$ Let $\hat{f}$ be in $C_0^\infty(\mathcal{O}')$ and let Ψ be a diffeomorphism from $\mathcal{O}$ to $\mathcal{O}'$ such that the Jacobians $D\Psi$ and $D\Psi^{-1}$ are bounded on $\mathcal{O}$ and $\mathcal{O}'$ respectively, then*

$$\|(\hat{f}\circ\Psi)^\vee\|_{L^2_{\delta+1}} \le C\|f\|_{L^2_{\delta+1}} \quad .$$

The constant C depends on Ψ, but is independent of f.

The proof of Proposition 4.1 now proceeds as follows. Let

$$g_j(x) = [(\chi_j\hat{f})(\xi(\eta))]^\vee(x) \quad ,$$

then it follows immediately from the formula for the w_j that

$$w_j(x) = \frac{1}{s}\left[\frac{\chi_j\hat{f}(\xi(\eta))}{\eta_j + i\eta_1}\right]^\vee(x) = \frac{1}{s}\left[\frac{\hat{g}_j(\eta)}{\eta_j + i\eta_1}\right]^\vee(x) \quad .$$

Using Lemma 4.1 we obtain that

$$\|w_j\|_{L^2_\delta} \le \frac{C}{s}\|g_j\|_{L^2_{\delta+1}} \quad . \tag{4.30}$$

At the same time, if we define

$$h_j = \left[\chi_j\hat{f}\right]^\vee(x) \quad \text{and} \quad \Psi(\eta) = \xi(\eta) \ ,$$

then we get

$$g_j = \left[\chi_j\hat{f}\circ\Psi\right]^\vee = \left[\hat{h}_j\circ\Psi\right]^\vee \quad ,$$

so that according to Lemma 4.2 and Lemma 4.3

$$\begin{aligned} \|g_j\|_{L^2_{\delta+1}} &\le C\|h_j\|_{L^2_{\delta+1}} = C\|(\chi_j\hat{f})^\vee\|_{L^2_{\delta+1}} \\ &\le C\|f\|_{L^2_{\delta+1}} \quad . \end{aligned} \tag{4.31}$$

A combination of (4.30) and (4.31) gives estimate

$$\|w_j\|_{L^2_\delta} \le \frac{C}{s}\|f\|_{L^2_{\delta+1}} \qquad s = \frac{|\zeta|}{\sqrt{2}} \ .$$

Envoking the formula $w = \sum_{j=1}^{n} w_j$ this completes the proof of Proposition 4.1. $\square$

It still remains to prove the three auxiliary lemmas 4.1–4.3. If we note that $\|\hat{f}\|_{H^{\delta+1}} = \|f\|_{L^2_{\delta+1}}$ then lemmas 4.2 and 4.3 merely state that multiplication by smooth, compactly supported functions and composition with smooth diffeomorphisms are bounded operators on $H^s(\mathbf{R}^n)$; two facts that are well known. It thus only remains to give the

Proof of Lemma 4.1 The map

$$f \mapsto \left(\frac{\hat{f}}{\xi_j + i\xi_1} \right)^{\vee}$$

may also be written

$$f \mapsto f * \left(\frac{1}{\xi_j + i\xi_1} \right)^{\vee} \qquad f \in \mathcal{S}(\mathbf{R}^n) \quad .$$

Furthermore it is well known that for an appropriate constant C

$$\left(\frac{1}{\xi_j + i\xi_1} \right)^{\vee} = \frac{C}{x_j + ix_1} \delta_0(\tilde{x}) \quad ,$$

where $\tilde{x} = (x_2, \ldots x_{j-1}, x_{j+1}, \ldots x_n)$ and δ_0 denotes a Dirac delta function at the origin. Therefore Z_j is given by

$$f \mapsto f * \frac{C}{x_j + ix_1} \delta_0(\tilde{x}) \quad ,$$

which is to say that Z_j is proportional to the solution operator for the inhomogeneous equation

$$(\partial_{x_1} - i\partial_{x_j})v = f \qquad \text{in } \mathbf{R}^n.$$

To prove Lemma 4.1 it clearly suffices to consider a single value of the index j, for instance $j = 2$. We furthermore claim that it suffices to prove the estimate $\|Z_2 f\|_{L^2_\delta} \le C\|f\|_{L^2_{\delta+1}}$ in $\mathbf{R}^2$. To see this we note that

$$\begin{aligned} \|u\|^2_{L^2_\delta(\mathbf{R}^n)} &= \int_{\mathbf{R}^n} (1 + |x|^2)^\delta |u(x)|^2 \, dx \\ &\le \int_{\mathbf{R}^n} (1 + x_1^2 + x_2^2)^\delta |u(x)|^2 \, dx \quad , \end{aligned}$$

since $\delta < 0$. Therefore

$$\|Z_2 f\|^2_{L^2_\delta(\mathbf{R}^n)} \le \int dx_3 \ldots dx_n \left[\|Z_2 f(\cdot, \cdot, x_3, \ldots x_n)\|^2_{L^2_\delta(\mathbf{R}^2)} \right] \quad . \tag{4.32}$$

Here we use the fact that $(Z_2f)(x_1, x_2, \dots x_n) = [Z_2f(\cdot, \tilde{x})](x_1, x_2)$, i.e., we use that $\tilde{x} = (x_3, \dots x_n)$ may be treated as parameters untouched by Z_2. At the same time

$$\begin{aligned}\|f\|^2_{L^2_{1+\delta}(\mathbf{R}^n)} &= \int_{\mathbf{R}^n} (1+|x|^2)^{1+\delta}|f(x)|^2\, dx \\ &\geq \int_{\mathbf{R}^n} (1+x_1^2+x_2^2)^{1+\delta}|f(x)|^2\, dx \quad ,\end{aligned}$$

since $1+\delta > 0$. Therefore

$$\|f\|^2_{L^2_{\delta+1}(\mathbf{R}^n)} \geq \int dx_3 \dots dx_n \left[\|f(\cdot,\cdot,x_3,\dots x_n)\|^2_{L^2_\delta(\mathbf{R}^2)}\right]. \tag{4.33}$$

The estimates (4.32) and (4.33) immediately imply that it suffices to prove the estimate $\|Z_2f\|_{L^2_\delta} \leq C\|f\|_{L^2_{\delta+1}}$ in two dimensions. This latter estimate follows from the following lemma with $p = 2$. □

Lemma 4.5. *If Z is defined by*

$$Zf := \int_{\mathbf{R}^2} \frac{1}{(u_2-v_2)+i(u_1-v_1)} f(v_1,v_2)\, dv \qquad f \in \mathcal{S}(\mathbf{R}^2) \quad ,$$

then Zf is bounded from $L^p_{\delta+1}(\mathbf{R}^2)$ to $L^p_\delta(\mathbf{R}^2)$ provided $p > 1$ and $-2/p < \delta < 1-2/p$

Proof

The space L^p_δ consists of the functions

$$\{u:\ (1+|x|^2)^{\delta/2}u \in L^p\} \ ,$$

equipped with the norm $\|u\|_{L^p_\delta} = \|(1+|x|^2)^{\delta/2}u\|_{L^p}$. It is well known that the spaces $L^q_{-\delta}$ and L^p_δ are dual, provided $1/p+1/q = 1$. Due to this fact, it suffices to verify the estimate $|\langle g, Zf\rangle| \leq C\|f\|_{L^p_{\delta+1}}\|g\|_{L^q_{-\delta}}$ for any $g \in L^q_{-\delta}$. We have

$$\begin{aligned}|\langle g, Zf\rangle| &= \left|\int_{\mathbf{R}^2}\int_{\mathbf{R}^2} \frac{g(u)f(v)}{(u_2-v_2)+i(u_1-v_1)}\, dudv\right| \\ &\leq \int_{\mathbf{R}^2}\int_{\mathbf{R}^2} \frac{\big(|g(u)|(1+|u|)^{\beta}(1+|v|)^{-\alpha}\big)\cdot\big(|f(v)|(1+|u|)^{-\beta}(1+|v|)^{\alpha}\big)}{|u-v|}\, dudv \quad ,\end{aligned}$$

where α and β will be chosen later. Employing Hölder's inequality,

$$\begin{aligned}|\langle g, Zf\rangle| \leq &\left(\int_{\mathbf{R}^2}\left\{\int_{\mathbf{R}^2} \frac{(1+|u|)^{-p\beta}(1+|v|)^{p(\alpha-\delta-1)}}{|u-v|}du\right\}(1+|v|)^{p(\delta+1)}|f(v)|^p dv\right)^{1/p} \\ &\times\left(\int_{\mathbf{R}^2}\left\{\int_{\mathbf{R}^2} \frac{(1+|u|)^{q(\beta+\delta)}(1+|v|)^{-q\alpha}}{|u-v|}dv\right\}(1+|u|)^{-q\delta}|g(u)|^q du\right)^{1/q} \\ \leq &\ C\|f\|_{L^p_{\delta+1}}\cdot\|g\|_{L^q_{-\delta}}\end{aligned}$$

with the constant C given by

$$C = \left(\sup_v \int_{\mathbf{R}^2} \frac{(1+|u|)^{-p\beta}(1+|v|)^{p(\alpha-\delta-1)}}{|u-v|} du \right)^{1/p}$$
$$\times \left(\sup_u \int_{\mathbf{R}^2} \frac{(1+|u|)^{q(\beta+\delta)}(1+|v|)^{-q\alpha}}{|u-v|} dv \right)^{1/q} .$$

In order to guarantee that C is finite, it suffices to require that

(4.34) $$1/p < \beta < 2/p \quad \text{and} \quad 1/q < \alpha < 2/q$$

with

(4.35) $$\delta = \alpha - \beta - 1/q \ .$$

On the other hand, if $p > 1$ and δ satisfies

$$-2/p < \delta < 1 - 2/p \ ,$$

then it is not difficult to check that it is always possible to select α and β such that (4.34) and (4.35) are satisfied. This completes the proof of Lemma 4.5 and consequently the proof of Lemma 4.1. □

§5. Identification of an Isotropic Conductivity in the Interior, $n \geq 3$

In the first part of this chapter we use the special solutions constructed in Chapter 4 together with the boundary identifiability result of Chapter 2 to prove a global identifiability result for dimension $n \geq 3$. This rsult is originally due to Sylvester and Uhlmann ([S-U II]). The case $n = 2$ will be considered in Chapter 6. In the second part of this chapter we extend the main ideas of the proof of the identifiability result in order to establish a result concerning the stable dependence of the conductivity on the boundary measurements. The main identifiability result is

Theorem 5.1 . *Let Ω be bounded domain in $\mathbf{R}^n (n \geq 3)$ with a smooth boundary and suppose that γ_1 and γ_2 are $C^\infty(\overline{\Omega})$ isotropic conductivities. If the two conductivities have the same Dirichlet- to Neumann-data map, i.e., if*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$$

then

$$\gamma_1 = \gamma_2$$

We will obtain Theorem 5.1 as a corollary of the analogous theorem for the Schroedinger operator (Theorem 5.2 below). Since the Dirichlet problem for the Schroedinger equation need not always have a (unique) solution, the Dirichlet- to Neumann-data map may not exist. It is quite natural to use the Cauchy data $\mathcal{C}_q$ in place of the Dirichlet- to Neumann-data map. The Cauchy data is defined as follows

$$\mathcal{C}_q = \{(f, g) \in H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega) : \\ \exists v \in H^1(\Omega) \text{ with } \Delta v + qv = 0 \text{ in } \Omega, \text{ and } v|_{\partial\Omega} = f, \ \frac{\partial v}{\partial \nu}|_{\partial\Omega} = g\} \ .$$

Theorem 5.2 . *Let Ω be a bounded domain in $\mathbf{R}^n (n \geq 3)$ with a smooth boundary and suppose that q_1 and q_2 are $L^\infty(\Omega)$ potentials such that*

$$\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$$

then

$$q_1 = q_2$$

Before giving the proof of Theorem 5.2, we show that Theorem 5.1 is its corollary.

Proof of Theorem 5.1. According to Theorem 5.1, γ_1 and γ_2 as well as $\frac{\partial \gamma_1}{\partial \nu}$ and $\frac{\partial \gamma_2}{\partial \nu}$ agree at $\partial\Omega$. Theorem 1.9. now guarantees that

$$\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$$

where q_1 and q_2 are defined by

$$q_j = -\frac{\Delta \gamma_j^{\frac{1}{2}}}{\gamma_j^{\frac{1}{2}}} \quad .$$

Thus Theorem 5.2 implies that $q_1 = q_2 = q_*$. Since $\Delta\gamma_j^{\frac{1}{2}} = -q_*\gamma_j^{\frac{1}{2}}$ and since $\gamma_1^{\frac{1}{2}} = \gamma_2^{\frac{1}{2}}$ and $\partial\gamma_1^{\frac{1}{2}}/\partial\nu = \partial\gamma_2^{\frac{1}{2}}/\partial\nu$ on $\partial\Omega$, it follows by unique continuation (ref) that $\gamma_1^{\frac{1}{2}} = \gamma_2^{\frac{1}{2}}$ in Ω; therefore $\gamma_1 = \gamma_2$ in Ω. In place of using this latter argument of unique continuation one might alternatively (as done in [A II], [S-U II]) consider the function

$$v = \log\left(\frac{\gamma_1}{\gamma_2}\right) \quad .$$

This function satisfies

$$\begin{aligned} &\nabla \cdot ((\gamma_1\gamma_2)^{\frac{1}{2}} \nabla v) = (\gamma_1\gamma_2)^{\frac{1}{2}}(q_1 - q_2) = 0 \\ &v|_{\partial\Omega} = 0 \end{aligned} \tag{5.1}$$

and hence

$$v \equiv 0 \text{ in } \Omega \ ,$$

i.e., $\gamma_1 = \gamma_2$ in Ω. □

We now turn to the proof of Theorem 5.2.

Proof of Theorem 5.2. Let $u_j \in H^1(\Omega)$, $j = 1, 2$ denote any pair of solutions to

$$\Delta u_j + q_j u_j = 0 \text{ in } \Omega \ .$$

We then have

$$\begin{aligned} \int_\Omega (q_1 - q_2) u_1 u_2 \ dx &= -\int_\Omega (\Delta u_1 u_2 - \Delta u_2 u_1) \ dx \\ &= -\int_{\partial\Omega} \left(\frac{\partial u_1}{\partial \nu} u_2 - \frac{\partial u_2}{\partial \nu} u_1\right) \ ds \quad . \end{aligned} \tag{5.2}$$

Since by assumption $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$ there exists a function $v \in H^1(\Omega)$ so that

$$\Delta v + q_1 v = 0 \text{ in } \Omega \ ,$$
$$v = u_2 \text{ and } \frac{\partial v}{\partial \nu} = \frac{\partial u_2}{\partial \nu} \text{ on } \partial\Omega \ .$$

The same calculation that led to (5.2) now gives

$$\begin{aligned} 0 = \int_\Omega (q_1 - q_1) u_1 v \ dx &= -\int_\Omega (\Delta u_1 v - \Delta v u_1) \ dx \\ &= -\int_{\partial\Omega} \left(\frac{\partial u_1}{\partial \nu} v - \frac{\partial v}{\partial \nu} u_1 \right) \ ds \\ &= -\int_{\partial\Omega} \left(\frac{\partial u_1}{\partial \nu} u_2 - \frac{\partial u_2}{\partial \nu} u_1 \right) \ ds \quad . \end{aligned} \tag{5.3}$$

A combination of (5.2) and (5.3) immediately yields that

$$\int_\Omega (q_1 - q_2) u_1 u_2 \ dx = 0 \tag{5.4}$$

for any pair of solutions to $\Delta u_j + q_j u_j = 0$.

At this point we shall choose u_1 and u_2 to be the type of solutions constructed in Theorem 4.3 of the previous chapter. Given any $k \in \mathbf{R}^n$ we choose ζ_1 and ζ_2 as follows:

$$\zeta_1 = l + i(k/2 + m)$$

$$\zeta_2 = -l + i(k/2 - m)$$

where l and m are required to satisfy

$$\begin{aligned} l \cdot k = l \cdot m &= k \cdot m = 0 \\ |m|^2 = |l|^2 &- \frac{|k|^2}{4} > 0 \\ |l| > \frac{1}{\epsilon} &\max \|(1 + |x|^2)^{\frac{1}{2}} q_j\|_{L^\infty} \ . \end{aligned} \tag{5.5}$$

The constant ϵ is the same as appears in Theorem 4.3 . It follows immediately that the ζ_j satify $\zeta_j \cdot \zeta_j = 0$, as well as

$$\frac{\|(1 + |x|^2)^{\frac{1}{2}} q_j\|_{L^\infty}}{|\zeta_j|} < \epsilon \ .$$

We also have that $|\zeta_j| \to \infty$ as $|l| \to \infty$. It is clear that the orthogonality relations required of l and m can only be satisfied if the dimension, n is three or higher. The solutions corresponding to the vectors ζ_j constructed in Theorem 4.3 have the form

$$u_j = e^{x \cdot \zeta_j} (1 + \psi_j(x, \zeta_j)) \ ,$$

where ψ_j satisfies

$$\|\psi_j\|_{L^2(\Omega)} \le \frac{C}{|\zeta_j|}\|q_j\|_{L^2(\Omega)} \ , \ \text{and} \ \|\psi_j\|_{H^1(\Omega)} \le C\|q_j\|_{L^2(\Omega)} \ . \tag{5.6}$$

The identity (5.4) therefore becomes

$$\begin{aligned} 0 &= \int_\Omega (q_1 - q_2)e^{x\cdot(\zeta_1+\zeta_2)}(1+\psi_1(x,\zeta_1))(1+\psi_2(x,\zeta_2))\ dx \\ &= \int_\Omega (q_1 - q_2)e^{x\cdot k}(1+\psi_1(x,\zeta_1))(1+\psi_2(x,\zeta_2))\ dx \ , \end{aligned}$$

which leads to

$$\int_\Omega e^{ix\cdot k}(q_1 - q_2)\ dx = -\int_\Omega e^{ix\cdot k}(q_1 - q_2)(\psi_1 + \psi_2 + \psi_1\psi_2)\ dx \quad . \tag{5.7}$$

The right hand side of (5.7) depends on l and m while the left hand side does not. If we let $|l|$ and hence $|m|$ approach infinity and use the first estimate of (5.6), then (5.7) becomes

$$\int_\Omega e^{ix\cdot k}(q_1 - q_2)\ dx = 0 \tag{5.8}$$

Let $\tilde{q}_i$ denote the function on all of $\mathbf{R}^n$ obtained by extending q_i to zero outside Ω. As k is arbitrary, and since the functions $\tilde{q}_i$ vanish outside Ω, (5.8) implies that the Fourier Transform of $(\tilde{q}_1 - \tilde{q}_2)$ vanishes identically. Therefore $\tilde{q}_1$ equals $\tilde{q}_2$; as a consequence q_1 equals q_2 in Ω and we are done. □

A somewhat more carefully crafted version of the uniqueness proof can be used to prove the stable dependence of γ on Λ_γ. By stability, or stable dependence, as opposed to continuous dependence, we mean that, under the hypothesis of an *à priori* bound for γ_1 and γ_2 (or q_1 and q_2) in a high norm, we can estimate the difference, $\gamma_1 - \gamma_2$ (or $q_1 - q_2$), in a lower norm in terms of the difference of the Dirichlet- to Neumann-data maps (or the Cauchy data). The stable dependence results presented here are except for minor modifications due to Alessandrini ([A II]). To measure the distance between the Dirichlet-to Neumann-data maps we use the operator norm for bounded operators between $H^{1/2}$ and $H^{-1/2}$. To measure the distance between the spaces of Cauchy data we use

$$\begin{aligned} \text{dist } (\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) = \max\Bigg\{ &\max_{(f,g)\in\mathcal{C}_{q_1}} \min_{(\tilde f,\tilde g)\in\mathcal{C}_{q_2}} \frac{\|(f,g)-(\tilde f,\tilde g)\|_{H^{1/2}\oplus H^{-1/2}}}{\|(f,g)\|_{H^{1/2}\oplus H^{-1/2}}}, \\ &\max_{(f,g)\in\mathcal{C}_{q_2}} \min_{(\tilde f,\tilde g)\in\mathcal{C}_{q_1}} \frac{\|(f,g)-(\tilde f,\tilde g)\|_{H^{1/2}\oplus H^{-1/2}}}{\|(f,g)\|_{H^{1/2}\oplus H^{-1/2}}} \Bigg\} \ . \end{aligned}$$

The norm on the space $H^{1/2} \oplus H^{-1/2}$ is defined by the expression

$$\|(f,g)\|_{H^{1/2}\oplus H^{-1/2}} = (\|f\|^2_{H^{1/2}} + \|g\|^2_{H^{-1/2}})^{1/2} \ .$$

It is not difficult to see that if the spaces $\mathcal{C}_{q_j}$ are both graphs of corresponding Dirichlet-to Neumann-data maps Λ_{q_j}, then one has the estimates

$$\begin{aligned} (1+\|\Lambda_{q_1}\|^2_{\frac12,-\frac12})^{-1/2}(1+\|\Lambda_{q_2}\|^2_{\frac12,-\frac12})^{-1/2}\|\Lambda_{q_1}-\Lambda_{q_2}\|_{\frac12,-\frac12} \\ \le \text{ dist } (\mathcal{C}_{q_1},\mathcal{C}_{q_2}) \le \|\Lambda_{q_1}-\Lambda_{q_2}\|_{\frac12,-\frac12} \ . \end{aligned} \tag{5.9}$$

We first show

Theorem 5.3. *Suppose that $s > \frac{n}{2}$, $n \ge 3$ and*

$$\|q_j\|_{H^s(\Omega)} \le M \tag{5.10}$$

then there exists $C = C(M)$ and $0 < \sigma < 1$ ($\sigma = \sigma(n)$) such that

$$\|q_1 - q_2\|_{H^{-1}(\Omega)} \le C\left(|\log\{ \text{ dist } (\mathcal{C}_{q_1},\mathcal{C}_{q_2})\}|^{-\sigma} + \text{ dist } (\mathcal{C}_{q_1},\mathcal{C}_{q_2})\right) \tag{5.11}$$

Proof. Our point of departure is the identity (5.2). If (f,g) is an arbitrary element of $\mathcal{C}_{q_1}$ then we have

$$\begin{aligned} \int_\Omega (q_1-q_2)u_1u_2\ dx &= -\int_{\partial\Omega}\left(\frac{\partial u_1}{\partial\nu}u_2 - \frac{\partial u_2}{\partial\nu}u_1\right)\ ds \\ &= -\int_{\partial\Omega}\left(\frac{\partial u_1}{\partial\nu}(u_2-f) - (\frac{\partial u_2}{\partial\nu}-g)u_1\right)\ dx \ . \end{aligned} \tag{5.12}$$

To obtain the last identity we have used integration by parts and the fact that there exists a function $v \in H^1(\Omega)$ such that

$$\Delta v + q_1 v = 0 \text{ in } \Omega \ ,$$
$$v = f \text{ and } \frac{\partial v}{\partial\nu} = g \text{ on } \partial\Omega \ .$$

We continue with

$$\begin{aligned} |\int_\Omega (q_1-q_2)u_1u_2\ dx| &\le \|\frac{\partial u_1}{\partial\nu}\|_{H^{-\frac12}}\|u_2-f\|_{H^{\frac12}} + \|u_1\|_{H^{\frac12}}\|\frac{\partial u_2}{\partial\nu}-g\|_{H^{-\frac12}} \\ &\le \|(u_1,\ \frac{\partial u_1}{\partial\nu})\|_{H^{\frac12}\oplus H^{-\frac12}}\cdot\|(u_2-f,\ \frac{\partial u_2}{\partial\nu}-g)\|_{H^{\frac12}\oplus H^{-\frac12}} \ . \end{aligned}$$

Now, minimization over all pairs (f,g) in $\mathcal{C}_{q_1}$ yields the estimate

$$(5.13)\quad |\int_\Omega (q_1-q_2)u_1u_2\,dx| \le \|(u_1,\ \frac{\partial u_1}{\partial\nu})\|_{H^{\frac12}\oplus H^{-\frac12}}\cdot \text{ dist }(\mathcal{C}_{q_1},\mathcal{C}_{q_2})\cdot \|(u_2,\ \frac{\partial u_2}{\partial\nu})\|_{H^{\frac12}\oplus H^{-\frac12}}$$

In case $\mathcal{C}_{q_1}$ and $\mathcal{C}_{q_2}$ are actually the graphs of Dirichlet- to Neumann-data maps Λ_{q_1} and Λ_{q_2}, an estimate corresponding to (5.13) would be

$$\begin{aligned}&\int_\Omega (q_1-q_2)u_1u_2\,dx\\ &\le \|u_1\|_{H^{\frac12}}\cdot(1+\|\Lambda_{q_1}\|^2_{\frac12,-\frac12})^{\frac12}\cdot\|\Lambda_{q_1}-\Lambda_{q_2}\|_{\frac12,-\frac12}\cdot\|u_2\|_{H^{\frac12}}(1+\|\Lambda_{q_2}\|^2_{\frac12,-\frac12})^{\frac12}\end{aligned} .$$

Our next step is to choose u_1 and u_2 to be the solutions produced in Theorem 4.3. That is

$$(5.14)\qquad u_j = e^{x\cdot\zeta_j}(1+\psi_j(x,\zeta_j))$$

with

$$\zeta_1 = l + i\left(\frac{k}{2}+m\right) ,$$

$$\zeta_2 = -l + i\left(\frac{k}{2}-m\right) ,$$

where k is arbitrary and l and m satisfy the requirements (5.5). The functions ψ_j satisfy the estimates (5.6). Since $u_j\in H^1(\Omega)$ are solutions to $\Delta u_j + q_ju_j = 0$ (with q_j bounded in L^∞) it follows that

$$\|\frac{\partial u_j}{\partial\nu}\|_{H^{-\frac12}(\partial\Omega)} \le C\|u_j\|_{H^1(\Omega)} .$$

Using the estimates (5.6) we now get

$$\begin{aligned}\|(u_1,\frac{\partial u_1}{\partial\nu})\|_{H^{\frac12}\oplus H^{-\frac12}} &\le C\|u_1\|_{H^1(\Omega)}\\ &\le C\left(\|e^{x\cdot\zeta_1}\|_{H^1(\Omega)}\|1+\psi_1\|_{L^2(\Omega)} + \|e^{x\cdot\zeta_1}\|_{L^2(\Omega)}\cdot\|1+\psi_1\|_{H^1(\Omega)}\right) ,\\ &\le C|\zeta_1|e^{|\zeta_1|\cdot D} + Ce^{|\zeta_1|D}\end{aligned}$$

where D denotes the constant $D=\max_{x\in\Omega}|x|$. Thus, for any fixed $D_*>D$

$$(5.15)\qquad \|(u_1,\frac{\partial u_1}{\partial\nu})\|_{H^{\frac12}\oplus H^{-\frac12}} \le Ce^{D_*|\zeta_1|} ,$$

uniformly in $|\zeta_1|$, and similarly

$$(5.16)\qquad \|(u_2,\frac{\partial u_2}{\partial\nu})\|_{H^{\frac12}\oplus H^{-\frac12}} \le Ce^{D_*|\zeta_2|} ,$$

uniformly in $|\zeta_2|$. Let r denote the parameter $r = \left(\frac{|k|^2}{4} + |m|^2 + |l|^2\right)^{\frac{1}{2}} - |k|$. In terms of r we have that $|\zeta_1| = |\zeta_2| = |k| + r$. The parameter r must be sufficiently large, *i.e.*,

$$r \geq C >> 1 \ ,$$

but it is otherwise free. A combination of the estimates (5.13)–(5.16) now yields

$$\begin{aligned} &| \int_\Omega (q_1 - q_2) e^{ix\cdot k} \ dx| \\ &\qquad \leq Ce^{2D_*(|k|+r)} \text{dist } (\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) + \int_\Omega |(q_1 - q_2||\psi_1 + \psi_2 + \psi_1\psi_2| \ dx \ , \end{aligned}$$

or, by use of (5.6) and (5.10)

$$|\widehat{(\tilde{q}_1 - \tilde{q}_2)}(k)| \leq C \left(e^{2D_*(|k|+r)} \text{dist } (\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) + 2M \cdot \frac{1}{|k|+r} \right) \ ,$$

where, as before, $\tilde{q}_j$ denotes the extension of q_j by zero outside Ω. We therefore have

$$\begin{aligned} &\|\tilde{q}_1 - \tilde{q}_2\|^2_{H^{-1}(\mathbf{R}^n)} \\ &= \int_{\mathbf{R}^n} |\widehat{(\tilde{q}_1 - \tilde{q}_2)}(k)|^2 (1 + |k|^2)^{-1} \ dk \\ &\leq \int_{|k|<\rho} |\widehat{(\tilde{q}_1 - \tilde{q}_2)}(k)|^2 (1 + |k|^2)^{-1} \ dk + \int_{|k|>\rho} |\widehat{(\tilde{q}_1 - \tilde{q}_2)}(k)|^2 (1 + \rho^2)^{-1} \ dk \\ &\leq C\rho^n \left(e^{4D_*(\rho+r)} (\text{dist } (\mathcal{C}_{q_1}, \mathcal{C}_{q_2}))^2 + \frac{1}{r^2} \right) + \frac{1}{1+\rho^2} \|\tilde{q}_1 - \tilde{q}_2\|^2_{L^2(\Omega)} \\ &\leq C\rho^n e^{4D_*(\rho+r)} (\text{dist } (\mathcal{C}_{q_1}, \mathcal{C}_{q_2}))^2 + C\frac{\rho^n}{r^2} + \frac{C}{\rho^2} \end{aligned} \tag{5.17}$$

In order to make the last two terms in the final expression of (5.17) small and of the same magnitude ($1/\rho^2$) we choose

$$r = \rho^{\frac{n+2}{2}}, \text{ for } \rho >> 1 \ .$$

With this choice we also have $r > \rho$. Concerning the first term in the ultimate expression of (5.17) we get

$$\rho^n e^{4D_*(\rho+r)} (\text{dist } (\mathcal{C}_{q_1}, \mathcal{C}_{q_2}))^2 \leq Ce^{Kr} (\text{dist } (\mathcal{C}_{q_1}, \mathcal{C}_{q_2}))^2 \ , \tag{5.18}$$

uniformly in ρ, for any fixed constant $K > 8D_*$. If we now choose

$$\rho = \left(\frac{1}{K} |\log\{\text{dist } (\mathcal{C}_{q_1}, \mathcal{C}_{q_2})\}| \right)^{\frac{2}{n+2}} \ ,$$

then

$$r = \frac{1}{K} |\log\{\text{dist } (\mathcal{C}_{q_1}, \mathcal{C}_{q_2})\}| \ ,$$

and therefore

$$e^{Kr} \leq \text{dist } (\mathcal{C}_{q_1}, \mathcal{C}_{q_2})^{-1} \ \text{ for dist } (\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) < 1 \tag{5.19}$$

A combination of the estimates (5.18) and (5.19) gives

$$\rho^n e^{4D_*(\rho+r)} (\text{dist } (\mathcal{C}_{q_1}, \mathcal{C}_{q_2}))^2 \leq C \text{dist } (\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) \tag{5.20}$$

provided dist $(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) < 1$. Insertion of (5.20) and the definition of ρ (and r) into the ultimate expression of (5.17) yields the estimate

$$\|\tilde{q}_1 - \tilde{q}_2\|^2_{H^{-1}(\mathbf{R}^n)} \leq C \left(|\log\{\text{dist } (\mathcal{C}_{q_1}, \mathcal{C}_{q_2})\}|^{-\frac{4}{n+2}} + \text{dist } (\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) \right) \ , \tag{5.21}$$

for dist $(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) < 1$. The estimate (5.21) is trivially satisfied for dist $(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) > 1$ because of the assumption (5.10). This completes the proof of Theorem 5.3. □

We now proceed to transform Theorem 5.3 into an analogous theorem for the conductivity problem. In the same fashion as earlier seen for the interior identifiability theorem the proof of the interior stable dependence result makes use of the the continuous dependence result (Theorem 2.2) concerning the boundary values. Among other things the proof depends on the following lemma

Lemma 5.1. *Suppose that $s > \frac{n}{2}$ and that γ_1 and γ_2 are C^∞ conductivities on $\bar{\Omega} \subset \mathbf{R}^n$, satisfying*

i) $1/E \leq \gamma_j \leq E$

ii) $\|\gamma_j\|_{H^{s+2}(\Omega)} \leq E$.

Let q_1 and q_2 denote the potentials defined by

$$q_j = -\frac{\Delta \gamma_j^{\frac{1}{2}}}{\gamma_j^{\frac{1}{2}}} \ . \tag{5.22}$$

There exists $C = C(\Omega, E, n, s)$ and σ ($\sigma = \sigma(s)$) such that

$$\begin{aligned} \textit{dist } (\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) &\leq \|\Lambda_{q_1} - \Lambda_{q_2}\|_{\frac{1}{2},-\frac{1}{2}} \\ &\leq C \left(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|^{\sigma}_{\frac{1}{2},-\frac{1}{2}} + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}} \right) \ . \end{aligned} \tag{5.23}$$

Proof. Since q_j are related to the conductivities γ_j by means of (5.22) it follows that $\mathcal{C}_{q_j}$ are graphs of corresponding Dirichlet- to Neumann-data maps. The first inequality of (5.23) comes directly from (5.9). A simple calculation gives that

$$\Lambda_{q_j}\phi = \gamma_j^{-\frac{1}{2}}\left(\Lambda_{\gamma_j}(\gamma_j^{-\frac{1}{2}}\phi) + \frac{\partial\gamma_j^{\frac{1}{2}}}{\partial\nu}\phi\right) \quad \forall\phi\in H^{\frac{1}{2}}(\partial\Omega)\ ,$$

so that

$$\begin{aligned}
&\|(\Lambda_{q_1}-\Lambda_{q_2})\phi\|_{H^{-\frac{1}{2}}} \\
&\quad\le C\|\gamma_1^{-\frac{1}{2}} - \gamma_2^{-\frac{1}{2}}\|_{C^1(\partial\Omega)}\|\Lambda_{\gamma_1}(\gamma_1^{-\frac{1}{2}}\phi) + \frac{\partial\gamma_1^{\frac{1}{2}}}{\partial\nu}\phi\|_{H^{-\frac{1}{2}}} \\
&\qquad + C\|\gamma_2^{-\frac{1}{2}}\|_{C^1(\partial\Omega)}\Bigg(\|\Lambda_{\gamma_1}(\gamma_1^{-\frac{1}{2}}\phi) - \Lambda_{\gamma_2}(\gamma_2^{-\frac{1}{2}}\phi)\|_{H^{-\frac{1}{2}}} \\
&\qquad\qquad + \|\frac{\partial\gamma_1^{\frac{1}{2}}}{\partial\nu} - \frac{\partial\gamma_2^{\frac{1}{2}}}{\partial\nu}\|_{C^0(\partial\Omega)}\|\phi\|_{L^2}\Bigg)\ .
\end{aligned}$$

Under the assumptions i) and ii) this immediately leads to the estimate

$$\begin{aligned}
&\|(\Lambda_{q_1}-\Lambda_{q_2})\phi\|_{H^{-\frac{1}{2}}} \\
&\quad\le C\left(\|\gamma_1-\gamma_2\|_{C^1(\partial\Omega)} + \|\Lambda_{\gamma_1}(\gamma_1^{-\frac{1}{2}}\phi) - \Lambda_{\gamma_2}(\gamma_2^{-\frac{1}{2}}\phi)\|_{H^{-\frac{1}{2}}}\right)\|\phi\|_{H^{\frac{1}{2}}}\ .
\end{aligned} \tag{5.24}$$

In a similar fashion we may also bound

$$\begin{aligned}
&\|\Lambda_{\gamma_1}(\gamma_1^{-\frac{1}{2}}\phi)-\Lambda_{\gamma_2}(\gamma_2^{-\frac{1}{2}}\phi)\|_{H^{-\frac{1}{2}}} \\
&\quad\le \|\Lambda_{\gamma_1}(\gamma_1^{-\frac{1}{2}} - \gamma_2^{-\frac{1}{2}})\phi\|_{H^{-\frac{1}{2}}} + \|(\Lambda_{\gamma_1} - \Lambda_{\gamma_2})(\gamma_2^{-\frac{1}{2}}\phi)\|_{H^{-\frac{1}{2}}} \\
&\quad\le C(\|\gamma_1-\gamma_2\|_{C^1(\partial\Omega)} + \|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}})\|\phi\|_{H^{\frac{1}{2}}}\ .
\end{aligned}$$

Insertion of this into (5.24) yields

$$\begin{aligned}
&\|(\Lambda_{q_1}-\Lambda_{q_2})\phi\|_{H^{-\frac{1}{2}}} \\
&\quad\le C(\|\gamma_1-\gamma_2\|_{C^1(\partial\Omega)} + \|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}})\|\phi\|_{H^{\frac{1}{2}}}\ .
\end{aligned} \tag{5.25}$$

Since $s-\frac{1}{2} > \frac{n}{2} - \frac{1}{2} = \frac{n-1}{2}$ we may use Sobolev's imbedding theorem and the logarithmic convexity of the Sobolev norms to obtain

$$\begin{aligned}
\|\gamma_1-\gamma_2\|_{C^1(\partial\Omega)} &\le \|\gamma_1-\gamma_2\|_{H^{s+\frac{1}{2}}(\partial\Omega)} \\
&\le C(\|\gamma_1-\gamma_2\|_{L^2(\partial\Omega)})^{\frac{2}{2s+3}}(\|\gamma_1-\gamma_2\|_{H^{s+\frac{3}{2}}(\partial\Omega)})^{\frac{2s+1}{2s+3}} \\
&\le C(\|\gamma_1-\gamma_2\|_{L^2(\partial\Omega)})^{\frac{2}{2s+3}}\ .
\end{aligned}$$

To obtain the last inequality we have also used the trace estimate

$$\|\gamma_1 - \gamma_2\|_{H^{s+\frac{3}{2}}(\partial\Omega)} \le C\|\gamma_1 - \gamma_2\|_{H^{s+2}(\Omega)} \ .$$

It now follows from the first part of the continuous depence result on the boundary (the estimate (2.24)) that

$$\|\gamma_1 - \gamma_2\|_{C^1(\partial\Omega)} \le C\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}}^{\frac{2}{2s+3}} \ . \tag{5.26}$$

After insertion of (5.26) into (5.25) we obtain the desired estimate with $\sigma = \frac{2}{2s+3}$. □

The stable dependence result for the conductivity problem is

Theorem 5.4. *Suppose that* $s > \frac{n}{2}$, $n \ge 3$, *and that* γ_1 *and* γ_2 *are* C^∞ *conductivities on* $\bar{\Omega} \subset \mathbf{R}^n$, *satisfying:*

i) $1/E \le \gamma_j \le E$

ii) $\|\gamma_j\|_{H^{s+2}(\Omega)} \le E$,

then there exists $C = C(\Omega, E, n, s)$ *and* $0 < \sigma < 1$ $(\sigma = \sigma(n,s))$ *such that*

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \le C\left\{|\log\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}}|^{-\sigma} + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}}\right\} \tag{5.27}$$

Proof. In light of the bound ii) it clearly suffices to prove the estimate (5.27) for $\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}}$ sufficiently small ($<< 1$). The last term in right hand side of (5.27) is there to render the estimate trivially satisfied for $\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}}$ large.

Consider the function

$$v = \log\left(\frac{\gamma_1}{\gamma_2}\right) = \log(\gamma_1) - \log(\gamma_2) \ .$$

This function satisfies the boundary value problem

$$\nabla\cdot((\gamma_1\gamma_2)^{\frac{1}{2}}\nabla v) = (\gamma_1\gamma_2)^{\frac{1}{2}}(q_1 - q_2) \ \text{ in } \Omega$$
$$v|_{\partial\Omega} = \gamma_1 - \gamma_2 \ ,$$

and hence

$$\begin{aligned} \|\log\gamma_1 - \log\gamma_2\|_{H^1(\Omega)} &= \|v\|_{H^1(\Omega)} \\ &\le C\left(\|q_1 - q_2\|_{H^{-1}(\Omega)} + \|\log\gamma_1 - \log\gamma_2\|_{H^{\frac{1}{2}}(\partial\Omega)}\right) \ . \end{aligned} \tag{5.28}$$

A combination of the estimates (5.11) and (5.23) gives that

$$
\begin{aligned}
\|q_1 - q_2\|_{H^{-1}(\Omega)} &\le C|\log\{ \text{ dist } (\mathcal{C}_{q_1}, \mathcal{C}_{q_2})\}|^{-\sigma_1} \\
&\le C|\log\left\{\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|^{\sigma_2}_{\frac12,-\frac12}\right\}|^{-\sigma_1} \\
&\le C|\log\left\{\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac12,-\frac12}\right\}|^{-\sigma_1}
\end{aligned}
\tag{5.29}
$$

for $\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac12,-\frac12}$ sufficiently small. In view of Sobolev's imbedding theorem and the logarithmic convexity of the Sobolev norms we have

$$
\begin{aligned}
\|\log\gamma_1 - \log\gamma_2\|_{L^\infty(\Omega)} &\le C\|\log\gamma_1 - \log\gamma_2\|_{H^s(\Omega)} \\
&\le C\|\log\gamma_1 - \log\gamma_2\|^{\frac{s-1}{s+1}}_{H^{s+2}(\Omega)} \cdot \|\log\gamma_1 - \log\gamma_2\|^{\frac{2}{s+1}}_{H^1(\Omega)} \\
&\le C\|\log\gamma_1 - \log\gamma_2\|^{\frac{2}{s+1}}_{H^1(\Omega)} \ .
\end{aligned}
$$

We also have

$$
\begin{aligned}
\|\log\gamma_1 - \log\gamma_2\|_{H^{\frac12}(\partial\Omega)} &\le C\|\log\gamma_1 - \log\gamma_2\|^{\frac{2s+2}{2s+3}}_{L^2(\partial\Omega)}\|\log\gamma_1 - \log\gamma_2\|^{\frac{1}{2s+3}}_{H^{s+\frac32}(\partial\Omega)} \\
&\le C\|\log\gamma_1 - \log\gamma_2\|^{\frac{2s+2}{2s+3}}_{L^\infty(\partial\Omega)} \ .
\end{aligned}
$$

Together with (5.28) and (5.29) these two estimates give

$$
\begin{aligned}
&\|\log\gamma_1 - \log\gamma_2\|_{L^\infty(\Omega)} \\
&\quad\le C\left(\|q_1 - q_2\|_{H^{-1}(\Omega)} + \|\log\gamma_1 - \log\gamma_2\|_{H^{\frac12}(\partial\Omega)}\right)^{\frac{2}{s+2}} \\
&\quad\le C\left(|\log\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac12,-\frac12}|^{-\sigma_1} + \|\log\gamma_1 - \log\gamma_2\|^{\frac{2s+2}{2s+3}}_{L^\infty(\partial\Omega)}\right)^{\frac{2}{s+2}} \ ,
\end{aligned}
\tag{5.30}
$$

for $\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac12,-\frac12}$ sufficiently small. Now

$$
\log\gamma_1 - \log\gamma_2 = \left[\int_0^1 \frac{dt}{(1-t)\gamma_1 + t\gamma_2}\right] \cdot (\gamma_1 - \gamma_2) \ ,
$$

and, since by hypothesis

$$
\frac{1}{E} \le \gamma_j \le E \ ,
$$

it follows that

$$
\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \le E\|\log\gamma_1 - \log\gamma_2\|_{L^\infty(\Omega)} \ ,
$$

and

$$
\|\log\gamma_1 - \log\gamma_2\|_{L^\infty(\partial\Omega)} \le E\|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)} \ .
$$

Therefore (5.30) translates into

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq C \left(|\log \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}}|^{-\sigma_1} + \|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)}^{\frac{2s+2}{2s+3}} \right)^{\frac{2}{s+2}} , \tag{5.31}$$

for $\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}}$ sufficiently small. By combination with the boundary continuous dependence result (Theorem 2.2) the estimate (5.31) becomes

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq C |\log \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}}|^{-\frac{2\sigma_1}{s+2}} ,$$

for $\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}}$ sufficiently small. This completes the proof of the theorem. $\square$