Recovering a Potential from Partial Cauchy Data

Alexander L. Bukhgeim[*]
Sobolev Institute of Mathematics
4 Koptyug Avenue,
Novisibirsk, 630090, Russia

Gunther Uhlmann[†]
Department of Mathematics
University of Washington
Seattle, WA 98195, USA

Abstract

In this paper we prove in dimension $n \geq 3$ that knowledge of the Cauchy data for the Schrödinger equation measured on particular subsets of the boundary determines uniquely the potential.

0 Introduction and Statement of the Result

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 boundary. We denote by ν the unit-outer normal to $\partial\Omega$. Let

$$H_\Delta(\Omega) = \{u \in \mathcal{D}'(\Omega) \mid u \in L^2(\Omega),\ \Delta u \in L^2(\Omega)\};$$

$H_\Delta(\Omega)$ is a Hilbert space with the norm

$$\|u\|^2_{H_\Delta(\Omega)} = \|u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)}.$$

For $u \in H_\Delta(\Omega)$, we have $u|_{\partial\Omega} \in H^{-\frac{1}{2}}(\partial\Omega)$ and $\frac{\partial u}{\partial\nu}|_{\partial\Omega} \in H^{-\frac{3}{2}}(\partial\Omega)$ (see Section 1). We define the set of Cauchy data for $q \in L^\infty(\Omega)$ by

$$C_q = \left\{ \left(u|_{\partial\Omega}, \frac{\partial u}{\partial\nu}\Big|_{\partial\Omega} \right) \mid (\Delta - q)u = 0 \text{ on } \Omega,\ u \in H_\Delta(\Omega) \right\}.$$

---

[*] This work was done while Alexander Bukhgeim was visiting the University of Washington. Partially supported by CRDF (Grant RM2-2242), RFBR (Grants 99–01–00540, 00–15–96183).

[†] Partly supported by a grant from NSF.

We have $C_q \subset H^{-\frac{1}{2}}(\partial\Omega) \times H^{-\frac{3}{2}}(\partial\Omega)$.

If 0 is not a Dirichlet eigenvalue of $\Delta - q$ in Ω then C_q contains the graph of the Dirichlet-to-Neumann map Λ_q conventionally defined on $H^{1/2}(\partial\Omega)$ by the relation $\Lambda_q(f) = \frac{\partial u}{\partial\nu}|_{\partial\Omega}$, where $u \in H^1(\Omega)$ is a solution to the problem

$$(\Delta - q)u = 0 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = f;$$

i.e.,

$$\{(f, \Lambda_q(f)) \mid f \in H^{1/2}(\partial\Omega)\} \subset C_q.$$

It was shown in [SU] in dimension $n \geq 3$ that C_q determines uniquely the potential. In two dimensions this is known for generic potentials [SuU1] and for potentials of the form $q = \frac{\Delta\gamma}{\gamma}$ with $\gamma > 0$ [N]. We also know that in two dimensions C_q determines the L^∞-singularities of q ([SuU2]).

It is an open problem whether partial information of C_q still determines uniquely the potential. The only case known previous to this paper is that of real-analytic q. In this case it is proven [KV] that if Γ denotes an open subset of $\partial\Omega$ and we measure $\Lambda_q(f)|_\Gamma$ for all f supported in Γ then we can recover q uniquely. This result follows since in this case we can determine from Λ_q all derivatives of q on Γ.

Before stating our main result we establish some notation.

Fix $\xi \in S^{n-1} = \{\mathbb{R}^n, |\xi| = 1.\}$ We define

$$\partial\Omega_+(\xi) = \{x \in \partial\Omega \mid \langle\nu, \xi\rangle > 0\},$$
$$\partial\Omega_-(\xi) = \{x \in \partial\Omega \mid \langle\nu, \xi\rangle < 0\}$$

and for $\varepsilon > 0$

$$\partial\Omega_{+,\varepsilon}(\xi) = \{x \in \partial\Omega \mid \langle\nu, \xi\rangle > \varepsilon\},$$
$$\partial\Omega_{-,\varepsilon}(\xi) = \{x \in \partial\Omega \mid \langle\nu, \xi\rangle < \varepsilon\}.$$

Here and in the sequel we use the notation $\langle a, b\rangle := \sum_{i=1}^n a_i b_i$ for real and complex vectors a and b.

We also define the set of restricted Cauchy data

$$C_{q,\varepsilon} = \left\{ \left(u|_{\partial\Omega}, \frac{\partial u}{\partial\nu}\Big|_{\partial\Omega_{-,\varepsilon}(\xi)} \right) \Big| \, (\Delta - q)u = 0 \text{ in } \Omega, \; u \in H_\Delta(\Omega) \right\}.$$

Our main result is

Theorem 0.1 *Let $n \geq 3$ and $q_i \in L^\infty(\Omega)$, $i = 1, 2$. Given $\xi \in S^{n-1}$ and $\varepsilon > 0$, assume that $C_{q_1,\varepsilon} = C_{q_2,\varepsilon}$. Then $q_1 = q_2$.*

Theorem 0.1 has an immediate consequence in Electrical Impedance Tomography. Let $\gamma \in C^2(\overline{\Omega})$ be a strictly positive function on $\overline{\Omega}$. the equation for the potential in the interior, under the assumption of no sinks or sources of current in Ω, is

$$\operatorname{div}(\gamma \nabla u) = 0 \text{ in } \Omega, \quad u|_{\partial\Omega} = f. \tag{0.1}$$

The Dirichlet-to-Neumann map is defined in this case as follows:

$$\Lambda_\gamma(f) = \left(\gamma \frac{\partial u}{\partial \nu}\right)\Big|_{\partial\Omega}.$$

It is easy to see that Λ_γ extends to a bounded map

$$\Lambda_\gamma : H^{-\frac{1}{2}}(\partial\Omega) \longrightarrow H^{-\frac{3}{2}}(\partial\Omega).$$

As a direct consequence of Theorem 0.1 we prove

Corollary 0.2 *Let $\gamma_i \in C^2(\overline{\Omega})$, $i = 1, 2$, be strictly positive. Given $\xi \in S^{n-1}$ and $\varepsilon > 0$, assume that*

$$\Lambda_{\gamma_1}(f)|_{\partial\Omega_{-,\varepsilon}(\xi)} = \Lambda_{\gamma_2}(f)|_{\partial\Omega_{-,\varepsilon}(\xi)} \quad \forall f \in H^{-\frac{1}{2}}(\partial\Omega).$$

Then $\gamma_1 = \gamma_2$.

As far as we know, Theorem 0.1 (Corollary 0.2) is the first global uniqueness result for the Schrödinger equation (conductivity equation) in which the Cauchy data are given only on part of the boundary beyond the case of a real-analytic potential. In [GU] it is proven that one can determine the Radon transform of the potential on a plane by measuring the Cauchy data of certain approximate solutions of the Schrödinger equation on a neighborhood of the intersection of the plane with the boundary.

We briefly outline the main steps of the proof of Theorem 0.1.

The global uniqueness result of [SU] uses complex geometrical optics solutions in the whole Euclidean space. Namely, let $q \in L^\infty(\mathbb{R}^n)$ have compact support. Then for $\rho \in \mathbb{C}^n$, $\langle \rho, \rho \rangle = 0$, $|\rho|$ sufficiently large, one can construct solutions to

$$(\Delta - q)u_\rho = 0$$

of the form

$$u_\rho = e^{\langle x,\rho\rangle}(1+\psi_q(x,\rho)) \tag{0.2}$$

with

$$\|\psi_q(\,\cdot\,,\rho)\|_{H^s(\Omega)} \le \frac{C}{|\rho|^{1-s}}, \quad 0 \le s \le 2, \tag{0.3}$$

for some $C > 0$ independent of ρ.

The function $\psi_q(x,\rho)$ solves

$$\Delta_\rho \psi_q = q(1+\psi_q),$$

where

$$\Delta_\rho(u) = e^{-\langle x,\rho\rangle}\Delta(e^{\langle x,\rho\rangle}u).$$

A natural way to attack the problem of finding a potential from partial information of the Cauchy data is to construct solutions of the form (0.2) with $\psi_q = 0$ on part of the boundary. In this paper we show that this is possible for particular subsets of the boundary. More precisely we prove

Lemma 0.3 *Let $n \ge 2$. Let $\rho \in \mathbb{C}^n$ with $\langle \rho,\rho\rangle = 0$ and $\rho = \tau(\xi + i\eta)$ with $\xi,\eta \in S^{n-1}$. Suppose that $f(\cdot,\rho/|\rho|) \in W^{2,\infty}(\Omega)$ satisfies $\partial_\xi f = \partial_\eta f = 0$, where ∂_ξ denotes the directional derivative in the direction ξ. Then we can find solutions to $(\Delta - q)u = 0$ in Ω of the form*

$$u(x,\rho) = e^{\langle x,\rho\rangle}\Big(f\Big(x,\frac{\rho}{|\rho|}\Big) + \psi(x,\rho)\Big), \quad \psi|_{\partial\Omega_-(\xi)} = 0, \tag{0.4}$$

with

$$\|\psi(\,\cdot\,,\rho)\|_{L^2(\Omega)} \le \frac{C}{\tau}, \quad \tau \ge \tau_0, \tag{0.5}$$

for some $C > 0$ and $\tau_0 > 0$.

The construction of the complex geometrical optics solutions (0.4) and the proof of Theorem 0.1 is radically different from the one in [SU]. We use Carleman estimates for the operator Δ_ρ, which is not an elliptic operator if we consider the dependence on the parameter ρ, to construct the solutions and prove the main result.

The Carleman estimates we use are developed in Section 2. In Section 1 we discuss some preliminaries related to the set of Cauchy data. In Section 3 we prove the main results of the paper.

1 Traces of Functions in the Space $H_\Delta(\Omega)$ and Green's Formula

In this section we discuss some preliminaries about traces and Green's formula for the space $H_\Delta(\Omega)$ defined in the previous section.

We have $H^2(\Omega) \subset H_\Delta(\Omega)$, where $H^m(\Omega)$ denotes the standard Sobolev space. Hence the trace theorem for the space $H_\Delta(\Omega)$ must be weaker than that for $H^2(\Omega)$. We will use the following lemma:

Lemma 1.1 *(i) Suppose that $\partial\Omega \in C^2$. Then the trace map*

$$\gamma_j u = \frac{\partial^j u}{\partial \nu^j}\Big|_{\partial\Omega}, \quad j = 0, 1,$$

defined on $C^\infty(\bar\Omega)$ has an extension, again denoted by γ_j, which is continuous from $H_\Delta(\Omega)$ into $H^{-j-1/2}(\partial\Omega)$. Thus there exists a constant $C > 0$ such that

$$\|\gamma_j u\|_{H^{-j-\frac{1}{2}}(\partial\Omega)} \le C\|u\|_{H_\Delta(\Omega)}. \tag{1.1}$$

(ii) If we assume in addition that $\gamma_0 u \in H^{\frac{3}{2}}(\partial\Omega)$ then $u \in H^2(\Omega)$, $\gamma_1 u \in H^{\frac{1}{2}}(\partial\Omega)$, and

$$\|u\|_{H^2(\Omega)} \le C(\|u\|_{H_\Delta(\Omega)} + \|\gamma_0 u\|_{H^{\frac{3}{2}}(\partial\Omega)}),$$
$$\|\gamma_1 u\|_{H^{\frac{1}{2}}(\partial\Omega)} \le C(\|u\|_{H_\Delta(\Omega)} + \|\gamma_0 u\|_{H^{\frac{3}{2}}(\partial\Omega)})$$

for some constant $C > 0$.

The proof of this result is standard (see, for example, [LM]). We give it here for the sake of completeness.

Proof. *(i)* Let $u \in C^\infty(\bar\Omega)$. For $w \in H^{\frac{1}{2}}(\partial\Omega)$ we define $l(w) = \int_{\partial\Omega} u\bar w\, dS$, where dS denotes the surface measure. By the standard trace theorem for $H^2(\Omega)$ functions, there exists a function $v \in H^2(\Omega)$ such that

$$\gamma_0 v = 0, \quad \gamma_1 v = \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = w,$$

and

$$\|v\|_{H^2(\Omega)} \le C\|w\|_{H^{\frac{1}{2}}(\partial\Omega)}.$$

Then using Green's formula, we obtain

$$l(w)=\int_{\partial\Omega} u\overline{\frac{\partial v}{\partial \nu}}\,dS=\int_{\Omega}(u\Delta\bar{v}-\Delta u\bar{v})\,dx$$

and hence

$$|l(w)|\le \|u\|_{H_\Delta(\Omega)}\|v\|_{H^2(\Omega)}\le C\|u\|_{H_\Delta(\Omega)}\|w\|_{H^{\frac{1}{2}}(\partial\Omega)}$$

which proves that $l\in H^{-\frac{1}{2}}(\partial\Omega)$ and the map $\gamma_1 : u\mapsto l\in H^{-\frac{1}{2}}(\partial\Omega)$ is bounded on $C^\infty(\bar{\Omega})$ provided with the norm of $H_\Delta(\Omega)$. Since $C^\infty(\bar{\Omega})$ is dense in $H_\Delta(\Omega)$, γ_0 can therefore be extended to a continuous linear map of $H_\Delta(\Omega)$ into $H^{-\frac{1}{2}}(\partial\Omega)$:

$$\|\gamma_0 u\|_{H^{-\frac{1}{2}}(\partial\Omega)}\le C\|u\|_{H_\Delta(\Omega)},$$

so we have proved (1.1) for $j=0$. Now we consider the case $j=1$. Let $u\in C^\infty(\bar{\Omega})$ and $w\in H^{\frac{3}{2}}(\partial\Omega)$. We define

$$l(w)=\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\bar{w}\,dS.$$

Then again by the trace theorem there exists a function $v\in H^2(\Omega)$ such that

$$\gamma_0 v=w,\quad \gamma_1 v=\frac{\partial v}{\partial \nu}\Big|_{\partial\Omega}=0,$$

and

$$\|v\|_{H^2(\Omega)}\le C\|w\|_{H^{\frac{3}{2}}(\partial\Omega)}.$$

Using Green's formula, we obtain

$$l(w)=\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\bar{v}\,dS=\int_{\Omega}(\Delta u\bar{v}-u\Delta\bar{v})\,dx.$$

Therefore,

$$|l(w)|\le \|u\|_{H_\Delta(\Omega)}\|v\|_{H^2(\Omega)}\le C\|u\|_{H_\Delta(\Omega)}\|w\|_{H^{\frac{3}{2}}(\partial\Omega)}.$$

As in the previous case this estimate proves the estimate (1.1) for $j=1$.

(ii) Consider first the case $\gamma_0 u = 0$. Take $u \in C^2(\bar{\Omega})$ with $\gamma_0 u = 0$. We have

$$\int_{\partial\Omega} \left|\frac{\partial u}{\partial \nu}\right|^2 \leq C \int_\Omega |\Delta u|^2\, dx; \tag{1.2}$$

$$2\int_\Omega |\nabla u|^2\, dx = -2\int_\Omega u\Delta u\, dx \leq \int_\Omega (|u|^2 + |\Delta u|^2)\, dx = \|u\|^2_{H_\Delta(\Omega)}. \tag{1.3}$$

Since $\partial\Omega \in C^2$, Kadlec's formula (see [T, p. 340]) gives

$$\sum_{|\alpha|=2} \int_\Omega |\partial^\alpha u|^2\, dx \leq \int_\Omega |\Delta u|^2\, dx + C\int_{\partial\Omega} \left|\frac{\partial u}{\partial \nu}\right|^2 dS \tag{1.4}$$

for some constant $C > 0$. Combining (1.2)–(1.4), we find that

$$\|u\|_{H^2(\Omega)} \leq C\|u\|_{H_\Delta(\Omega)}, \quad u \in C^2(\bar{\Omega}), \quad \gamma_0 u = 0,$$

with another constant $C > 0$. Using standard density arguments, we obtain

$$\|u\|_{H^2(\Omega)} \leq C\|u\|_{H_\Delta(\Omega)} \quad \forall u \in H_\Delta(\Omega) \cap \{u \mid \gamma_0 u = 0\}.$$

Thus $u \in H^2(\Omega)$ and by the trace theorem $\gamma_1 u \in H^{\frac{1}{2}}(\partial\Omega)$ and

$$\|\gamma_1 u\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C\|u\|_{H^2(\Omega)} \leq C_1\|u\|_{H_\Delta(\Omega)}.$$

This result also follows from general regularity results near the boundary (see, for example, [T, pp. 393–394], where this type of result is stated for $\partial\Omega \in C^\infty$ and general elliptic operators; for the Laplacian it is sufficient to assume that $\partial\Omega \in C^2$). Thus we have proved *(ii)* in the case $\gamma_0 u = 0$.

If $\gamma_0 u \in H^{\frac{3}{2}}(\partial\Omega)$ by the standard trace theorem there exists $v \in H^2(\Omega)$ such that $\gamma_0 u = \gamma_0 v$ and then the function $w := u - v \in H_\Delta(\Omega)$ and $\gamma_0 w = 0$. Thus we have reduced the general case to the case $\gamma_0 u = 0$, completing the proof of the Lemma.

As a straightforward consequence we obtain the following

Corollary 1.2 *(i) For $u \in H_\Delta(\Omega)$ and $v \in H^2(\Omega)$ we have the generalized Green's formula*

$$\int_\Omega (\Delta - q)u\bar{v}\, dx = \int_\Omega u\overline{(\Delta - \bar{q})v}\, dx + \int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu}\bar{v} - u\frac{\overline{\partial v}}{\partial \nu}\right) dS, \tag{1.5}$$

where $q \in L^\infty(\Omega)$, $\partial\Omega \in C^2$, *and*

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu} \bar{v}\, dS := \langle \gamma_1 u, \gamma_0 \bar{v} \rangle$$

denotes the duality between $H^{-\frac{3}{2}}(\partial\Omega)$ *and* $H^{\frac{3}{2}}(\partial\Omega)$ *and*

$$\int_{\partial\Omega} u \overline{\frac{\partial v}{\partial \nu}}\, dS := \langle \gamma_0 u, \gamma_1 \bar{v} \rangle$$

denotes the duality between $H^{-\frac{1}{2}}(\partial\Omega)$ *and* $H^{\frac{1}{2}}(\partial\Omega)$.

(ii) We define $\overset{\circ}{H}_\Delta(\Omega) = \{u \in H_\Delta(\Omega) \mid \gamma_0 u = 0,\ \gamma_1 u = 0\}$. *Then (since* $H^2(\Omega)$ *is dense in* $H_\Delta(\Omega)$*) we have*

$$\int_\Omega (\Delta - q) u \bar{v}\, dx = \int_\Omega u \overline{(\Delta - \bar{q}) v}\, dx$$

for all $u \in \overset{\circ}{H}_\Delta(\Omega)$ *and* $v \in H_\Delta(\Omega)$.

2 Carleman Estimates

As before we fix $\xi \in S^{n-1}$. In this section we prove the following estimate:

Lemma 2.1 *For all complex-valued* $u \in C^2(\bar{\Omega})$, $u|_{\partial\Omega} = 0$, *and all* $\tau > 0$ *we have*

$$\begin{aligned} &\frac{8\tau^2}{d^2} \int_\Omega e^{-2\tau\langle x,\xi\rangle} |u|^2\, dx + \int_{\partial\Omega_+} 2\tau \langle \xi, \nu \rangle e^{-2\tau\langle x,\xi\rangle} |\partial_\nu u|^2\, dS \\ &\leq \int_\Omega e^{-2\tau\langle x,\xi\rangle} |\Delta u|^2\, dx - \int_{\partial\Omega_-} 2\tau \langle \xi, \nu \rangle e^{-2\tau\langle x,\xi\rangle} |\partial_\nu u|^2\, dS. \end{aligned} \tag{2.1}$$

Here $d = b - a$, where $\Omega \subset \{x \in \mathbb{R}^n \mid a < \langle x, \xi \rangle < b\}$.

Proof. In the estimate below we use the following notations:

$$\partial = \nabla = (\partial_{x_1}, ..., \partial_{x_n}), \quad \partial_\xi = \langle \nabla, \xi \rangle, \quad v_k = \frac{\partial v}{\partial x_k}, \quad v_{kj} = \frac{\partial^2 v}{\partial x_k x_j}.$$

We define

$$I = \int_\Omega e^{-2\tau\langle \xi, x\rangle} |\Delta u(x)|^2\, dx, \quad \tau \geq 0,$$

and estimate it from below. We have

$$I = \int_\Omega |e^{-\tau\langle\xi,x\rangle}\Delta e^{\tau\langle\xi,x\rangle}e^{-\tau\langle\xi,x\rangle}u|^2 = \int_\Omega |P_+v + P_-v|^2\,dx \tag{2.2}$$
$$= \int_\Omega (|P_+v|^2 + |P_-v|^2 + 2\operatorname{Re}(P_+v\,\overline{P_-v}))\,dx,$$

where $P = e^{-\tau\langle\xi,x\rangle}\Delta e^{\tau\langle\xi,x\rangle} = \Delta + 2\tau\partial_\xi + \tau^2$, $P_+ = \Delta + \tau^2$, $P_- = 2\tau\partial_\xi$, and

$$v = e^{-\tau\langle\xi,x\rangle}u. \tag{2.3}$$

A straightforward calculations give

$$2\operatorname{Re}(P_+v\,\overline{P_-v}) = 2\operatorname{Re}((\Delta+\tau^2)v\,2\tau\partial_\xi\bar v) = 2\operatorname{Re}(\Delta v\,2\tau\partial_\xi\bar v) + 4\tau^3\operatorname{Re}(v\,\partial_\xi\bar v).$$

We have $2\operatorname{Re}(v\,\partial_\xi\bar v) = \partial_\xi|v|^2$ and hence

$$4\tau^3\operatorname{Re}(v\,\partial_\xi\bar v) = 2\tau^3\partial_\xi|v|^2. \tag{2.4}$$

Since $2\operatorname{Re}(v_{kk}\,\bar v_j) = 2\operatorname{Re}(\partial_k(v_k\bar v_j)) - 2\operatorname{Re}(v_k\bar v_{kj}) = 2\partial_k\operatorname{Re}(v_k\bar v_j) - \partial_j|v_k|^2$, we conclude that

$$2\operatorname{Re}(\Delta v\,2\tau\partial_\xi\bar v) = (2\tau)2\operatorname{Re}\sum_{kj} v_{kk}\xi_j\bar v_j \tag{2.5}$$
$$= 2\tau\sum_k\Big(\partial_k 2\operatorname{Re}\sum_j \bar v_j v_k\xi_j - \sum_j \xi_j\partial_j|v_k|^2\Big)$$
$$= 2\tau(2\operatorname{div}(\operatorname{Re}(\partial_\xi\bar v\,\nabla v)) - \operatorname{div}(|\nabla v|^2\xi)).$$

So from (2.4) and (2.5) we obtain

$$2\operatorname{Re}(P_+v\,\overline{P_-v}) = \operatorname{div}(4\tau\operatorname{Re}(\partial_\xi\bar v\,\nabla v) - 2\tau|\nabla v|^2\xi + 2\tau^3|v|^2\xi)$$

and hence the divergence theorem gives

$$\int_\Omega 2\operatorname{Re}(P_+v\,\overline{P_-v})\,dx \tag{2.6}$$
$$= \int_{\partial\Omega} 4\tau\operatorname{Re}(\partial_\xi\bar v\,\partial_\nu v) - 2\tau\langle\nu,\xi\rangle|\nabla v|^2 + 2\tau^3\langle\nu,\xi\rangle|v|^2\,dS.$$

Since $u|_{\partial\Omega} = 0$, $v|_{\partial\Omega} = 0$ and therefore the tangential component of $\nabla v|_{\partial\Omega}$ is zero. Thus

$$|\nabla v|^2 = |\partial_\nu v|^2, \quad \partial_\xi v = \langle\xi,\nu\rangle\partial_\nu v. \tag{2.7}$$

Using (2.6) and (2.7), we conclude that

$$\begin{aligned}\int_\Omega 2\,\mathrm{Re}(P_+v\,\overline{P_-v})\,dx \\ = \int_{\partial\Omega} 2\tau\langle\xi,\nu\rangle|\partial_\nu v|^2\,dS = \int_{\partial\Omega} 2\tau e^{-2\tau\langle\xi,x\rangle}\langle\xi,\nu\rangle|\partial_\nu u|^2\,dS,\end{aligned} \tag{2.8}$$

because, by (2.3), $v_j = e^{-\tau\langle\xi,x\rangle}(u_j - \tau\xi_j u) = e^{-\tau\langle\xi,x\rangle}u_j$ on $\partial\Omega$, since $u|_{\partial\Omega} = 0$.

We also need Poincaré's inequality in the following form (see [DL vol. 2, pp. 125–126]).

Proposition 2.2 *Let Ω be an open set in $\mathbb{R}^n$ lying in a strip $\{x \in \mathbb{R}^n \mid a < \langle x,\xi\rangle < b\}$, and let $d = b - a$. Then for all $v \in H_0^2(\Omega)$ we have*

$$\int_\Omega |v|^2\,dx \le \frac{d^2}{2}\int_\Omega |\partial_\xi v|^2\,dx. \tag{2.9}$$

It follows from (2.9) that

$$\int_\Omega |P_-v|^2\,dx = 4\tau^2\int_\Omega |\partial_\xi v|^2\,dx \ge \frac{8\tau^2}{d^2}\int_\Omega |v|^2\,dx = \frac{8\tau^2}{d^2}\int_\Omega e^{-2\tau\langle x,\xi\rangle}|u|^2\,dx.$$

Then, appealing to (2.2), (2.8), and the last inequality, we complete the proof of Lemma 2.1.

By taking τ sufficiently large, we can derive the following estimate for the Schrödinger operator:

Corollary 2.3 *(i) For $q \in L^\infty(\Omega)$ there exist $\tau_0 > 0$ and $C > 0$ such that for all $u \in C^2(\bar\Omega)$, $u|_{\partial\Omega} = 0$, and $\tau \ge \tau_0$ we have the estimate*

$$\begin{aligned}C\tau^2\int_\Omega |e^{-\tau\langle x,\xi\rangle}u|^2\,dx + \tau\int_{\partial\Omega_+}\langle\xi,\nu\rangle|e^{-\tau\langle x,\xi\rangle}\partial_\nu u|^2\,dS \\ \le \int_\Omega |e^{-\tau\langle x,\xi\rangle}(\Delta - q)u|^2\,dx - \tau\int_{\partial\Omega_-}\langle\xi,\nu\rangle|e^{-\tau\langle x,\xi\rangle}\partial_\nu u|^2\,dS.\end{aligned} \tag{2.10}$$

(ii) Let $\Omega \subset \mathbb{R}^n$ be an arbitrary bounded open set. For all $u \in C_0^2(\Omega)$ and $\tau \ge \tau_0$ we have

$$C\tau^2\int_\Omega |e^{-\tau\langle x,\xi\rangle}u|^2\,dx \le \int_\Omega |e^{-\tau\langle x,\xi\rangle}(\Delta - q)u|^2\,dx.$$

Proof. For $q \in L^\infty(\Omega)$,

$$|\Delta u|^2 = |(\Delta - q)u + qu|^2 \leq 2|(\Delta - q)u|^2 + 2\|q\|^2_{L^\infty(\Omega)}|u|^2$$

and hence if we choose $\tau_0 > 0$ such that

$$0 < \frac{8}{d^2} - 2\|q\|^2_{L^\infty(\Omega)} \cdot \tau_0^{-2} =: 2C \tag{2.11}$$

then from (2.1) and (2.11) we obtain the corollary.

3 Proof of the Results

In this section we prove the results stated in the Introduction. We first prove Lemma 0.3.

Let $\xi \in S^{n-1}$ be fixed as in the previous sections. Let $L^2_\tau(\Omega)$, $\tau \in \mathbb{R}$, be the weighted Hilbert space with the scalar product $\langle u, v\rangle_\tau = \int_\Omega e^{2\tau\langle x,\xi\rangle} u(x)\bar{v}(x)\,dx$ and the norm $\|u\|^2_\tau = \int_\Omega e^{2\tau\langle x,\xi\rangle}|u(x)|^2\,dx$. It is clear that $L^2_\tau(\Omega)$ and $L^2_{-\tau}(\Omega)$ with pairing $\langle u, v\rangle_0$ are dual to each other and every bounded linear functional on $L^2_\tau(\Omega)$ has the form $l(u) = \langle u, v\rangle_0$, where $v \in L^2_{-\tau}(\Omega)$. We define

$$\mathcal{D} = \Big\{ v \in C^2(\bar{\Omega}) \;\Big|\; v|_{\partial\Omega} = 0,\ \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega_+(\xi)} = 0 \Big\}.$$

Let $f \in L^2_{-\tau}(\Omega)$. For $q \in L^\infty(\Omega)$, we define a linear functional l on the linear subspace $L = (\Delta - \bar{q})\mathcal{D}$ of $L^2_\tau(\Omega)$ as follows:

$$l((\Delta - \bar{q})v) = \langle v, f\rangle_0. \tag{3.1}$$

Using the Carleman estimate (2.10) with ξ replaced by $-\xi$ and u replaced by $v \in \mathcal{D}$ and noticing that $\partial\Omega_-(-\xi) = \partial\Omega_+(\xi)$, we obtain

$$C\tau^2\|v\|^2_\tau \leq \|(\Delta - \bar{q})v\|^2_\tau, \quad \tau \geq \tau_0 > 0, \tag{3.2}$$

with $C > 0$. Thus, using (3.1) and (3.2), we conclude that

$$|l((\Delta - \bar{q})v)| \leq \|v\|_\tau\, \|f\|_{-\tau} \leq \frac{1}{\tau\sqrt{C}}\|(\Delta - \bar{q})v\|_\tau \|f\|_{-\tau}.$$

Thus l is bounded on $L \subset L^2_\tau(\Omega)$ and, by the Hahn–Banach theorem, it has an extension l to a linear functional on $L^2_\tau(\Omega)$ with the same norm. Therefore there exists $u \in L^2_{-\tau}(\Omega)$ such that

$$\langle v, f\rangle_0 = \langle(\Delta - \bar{q})v, u\rangle_0 \tag{3.3}$$

and

$$\|u\|_{-\tau} = \|l\| \leq \frac{1}{\tau\sqrt{C}}\|f\|_{-\tau}, \qquad \tau \geq \tau_0 > 0.$$

¿From (3.3) we conclude that u is an $L^2(\Omega)$ solution to the equation $(\Delta - q)u = f$. Moreover, since $u, f \in L^2(\Omega)$ we have $u \in H_\Delta(\Omega)$. Green's formula (1.5) gives

$$\int_{\partial\Omega} u\frac{\partial\bar{v}}{\partial\nu}\, dS = 0 \qquad \forall\, v \in C^2(\bar{\Omega}),\ \frac{\partial v}{\partial\nu}\Big|_{\partial\Omega_+(\xi)} = 0$$

and hence $u = 0$ on $\partial\Omega_-(\xi)$. Thus we have proved

Lemma 3.1 *For every $f \in L^2_{-\tau}(\Omega)$, $\tau \geq \tau_0$, there is $u \in L^2_{-\tau}(\Omega)$ such that*

$$(\Delta - q)u = f \text{ on } \Omega, \quad u|_{\partial\Omega_-(\xi)} = 0.$$

Moreover,

$$\|u\|_{-\tau} \leq \frac{C}{\tau}\|f\|_{-\tau}, \quad \tau \geq \tau_0,$$

for some constant $C > 0$.

End of proof of Lemma 0.3. As in the statement of the Lemma 0.3 we define $\rho = \tau(\xi + i\eta)$, $\xi, \eta \in S^{n-1}$, $\langle\xi, \eta) = 0$, $\tau \geq \tau_0$. Then $\langle\rho, \rho\rangle = 0$, $|\rho|^2 = 2\tau^2$. Let $f(\cdot, \frac{\rho}{|\rho|})$ be a function in $W^{2,\infty}(\Omega)$ such that $\partial_\xi f = \partial_\eta f = 0$ on Ω (in the two dimensional case this implies that $f = \text{const}$).

We define $v = u - e^{\langle\rho,x\rangle}f(x, \frac{\rho}{|\rho|})$ and $\psi = e^{-\langle\rho,x\rangle}v$. Then we have

$$(\Delta - q)v = (qf - \Delta f)e^{\langle\rho,x\rangle} \in L^2_{-\tau}(\Omega), \quad v|_{\partial\Omega_-(\xi)} = 0.$$

Since

$$\|(qf - \Delta f)e^{\langle\rho,x\rangle}\|_{-\tau} = \|qf - \Delta f\|_0 \leq \|qf - \Delta f\|_{L^\infty}(\text{vol }\Omega)^{1/2},$$

using Lemma 3.1, we obtain

$$\|\psi\|_{L_2(\Omega)} = \|v\|_{-\tau} \leq \frac{C}{\tau}\|qf - \Delta f\|_{L_\infty}(\text{vol }\Omega)^{1/2} \to 0 \quad \text{as} \quad \tau \to \infty$$

for some constant $C > 0$.

Remark 3.2 *In Lemma 3.1 we assumed that* $\partial\Omega \in C^2$*, since we use Green's formula (1.5). But if we define in the proof of Lemma 3.1*

$$\mathcal{D} = C_0^2(\Omega)$$

then we will have the analog of Lemma 3.1 for an arbitrary open bounded domain $\Omega \subset \mathbb{R}^n$ *without boundary condition on* $\partial\Omega_-(\xi)$*. Namely, we have proved*

Lemma 3.3 *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *(with no regularity assumptions on the boundary). There exists a solution* $u \in H_\Delta(\Omega)$ *to* $(\Delta - q)u = 0$ *in* Ω *which has the form* (0.2) *and satisfies the estimate* (0.3) *with* $s = 0$.

Proof of Theorem 0.1. As before we let $\xi \in S^{n-1}$. Fix $k \in \mathbb{R}^n$ such that $\langle \xi, k\rangle = 0$. Using Lemma 0.3, we choose a solution $u_2 \in H_\Delta(\Omega)$ to $(\Delta - q_2)u_2 = 0$ in Ω of the form

$$u_2 = e^{\langle x,\rho_2\rangle}(1 + \psi_{q_2}(x,\rho_2))$$

as in (0.2) with

$$\rho_2 = \tau\xi - i\frac{k+l}{2},$$

where $\langle l,k\rangle = \langle l,\xi\rangle = 0$ and $|k+l|^2 = 4\tau^2$ (with these conditions $\langle \rho_2,\rho_2\rangle = 0$). In dimension $n \geq 3$ we can always choose such a vector l.

Since $C_{q_1,\varepsilon} = C_{q_2,\varepsilon}$, there is a solution $u_1 \in H_\Delta(\Omega)$ to $(\Delta - q_1)u_1 = 0$ in Ω such that

$$u_1|_{\partial\Omega} = u_2|_{\partial\Omega}, \quad \frac{\partial u_1}{\partial\nu}\Big|_{\partial\Omega_{-,\varepsilon}(\xi)} = \frac{\partial u_2}{\partial\nu}\Big|_{\partial\Omega_{-,\varepsilon}(\xi)}.$$

Let us denote $u := u_1 - u_2$ and $q := q_1 - q_2$. We have

$$(\Delta - q_1)u = qu_2 \text{ in } \Omega, \quad u|_{\partial\Omega} = 0.$$

Note that, since $u|_{\partial\Omega} = 0$ and $u \in H_\Delta(\Omega)$ (in both cases), from Lemma 1.1 (ii) we conclude that $u \in H^2(\Omega)$. Therefore, by Corollary 1.2, for an arbitrary $v \in H_\Delta(\Omega)$ we can use Green's formula (1.5) and obtain

$$\int_\Omega (\Delta - q_1)u\bar{v}\,dx = \int_\Omega qu_2\bar{v}\,dx = \int_\Omega u(\Delta - q_1)\bar{v}\,dx + \int_{\partial\Omega}\frac{\partial u}{\partial\nu}\bar{v}\,dS;$$

i.e.,

$$\int_\Omega q u_2 \bar{v}\, dx = \int_\Omega u(\Delta - q_1)\bar{v}\, dx + \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\bar{v}\, dS. \tag{3.4}$$

Now, we choose

$$\bar{v} = e^{\langle x,\rho_1\rangle}(1+\psi_{q_1}(x,\rho_1))$$

as in (0.2) to be a solution to $(\Delta - q_1)\bar{v} = 0$, where

$$\rho_1 = -\tau\xi - i\frac{k-l}{2}$$

with ξ, k, and l as before so that $\langle \rho_1, \rho_1\rangle = 0$. Notice that with this choice of ρ_j, $j = 1, 2$, we have

$$\rho_1 + \rho_2 = -ik.$$

With these choices of u_2 and v, the identity (3.4) now reads

$$\int_\Omega q u_2 \bar{v} = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\bar{v}\, dS. \tag{3.5}$$

The final step in the proof is to show that the right hand side of (3.5) goes to 0 as $\tau \to \infty$.

By hypothesis,

$$\partial_\nu u|_{\partial\Omega_{-,\varepsilon}(\xi)} = 0.$$

Then we have

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu}\bar{v}\, dS = \int_{\partial\Omega\setminus\partial\Omega_{-,\varepsilon}} \frac{\partial u}{\partial \nu}\bar{v}\, dS = \int_{\partial\Omega_{+,\varepsilon}} \frac{\partial u}{\partial \nu}\bar{v}\, dS.$$

The Cauchy–Schwarz inequality and the estimate $\|\psi_{q_1}\|_{C(\partial\Omega)} \le C\tau^{1/4}$, which follows from (0.3) and the Sobolev embedding theorem, yields

$$\begin{aligned}\left|\int_{\partial\Omega} \frac{\partial u}{\partial \nu}\bar{v}\, dS\right| &= \left|\int_{\partial\Omega_{+,\varepsilon}} \frac{\partial u}{\partial \nu} e^{\langle x,\rho_1\rangle}(1+\psi_{q_1}(x,\rho_1))\, dS\right| \\ &\le \int_{\langle \xi,\nu\rangle \ge \varepsilon} \left|\frac{\partial u}{\partial \nu} e^{-\tau\langle \xi,x\rangle}(1+\psi_{q_1}(\,\cdot\,,\rho_1))\right| dS \\ &\le C(1+\tau^{1/4})(\operatorname{vol}\partial\Omega_{+,\varepsilon})^{1/2}\left(\int_{\langle \xi,\nu\rangle\ge\varepsilon} |e^{-\tau\langle\xi,x\rangle}\partial_\nu u|^2\, dS\right)^{\frac{1}{2}}\end{aligned}$$

for some $C > 0$. Now we use the Carleman estimate (2.10) to obtain

$$\begin{aligned}\tau\varepsilon \int_{\partial\Omega_{+,\varepsilon}} |e^{-\tau\langle\xi,x\rangle}\partial_\nu u|^2 \, dS &\le \tau \int_{\partial\Omega_+} \langle\xi,x\rangle |e^{-\tau\langle\xi,x\rangle}\partial_\nu u|^2 \, dS \\ &\le \int_\Omega |e^{-\tau\langle\xi,x\rangle}(\Delta - q_1)u|^2 \, dx = \int_\Omega |e^{-\tau\langle\xi,x\rangle} q u_2|^2 \, dx \\ &\le 2(\|q_1\|_{L^\infty(\Omega)} + \|q_2\|_{L^\infty(\Omega)})^2 (1 + \|\psi_2\|^2_{L^2(\Omega)}).\end{aligned}$$

Hence, we have proved that

$$\left| \int_{\partial\Omega} \frac{\partial u}{\partial\nu} \bar{v} \, dS \right| \le C\tau^{-1/4} \to 0, \quad \tau \to \infty. \tag{3.7}$$

By letting $\tau \to \infty$ and using (0.3), (0.5), (3.5), and (3.7), we conclude that

$$\int_\Omega e^{-i\langle x,k\rangle} q(x) \, dx = 0$$

for all $k \perp \xi$. Changing $\xi \in S^{n-1}$ in a small conic neighborhood and using the fact that $\widehat{q}(k)$ is analytic we get that $q = 0$ finishing the proof of Theorem 0.1.

Proof of Corollary 0.2. As is well known by now we can transform the problem (0.1) to the Schrödinger equation using the transformation $w = \gamma^{\frac{1}{2}} u$. If u solves (0.1), then w solves

$$(\Delta - q)w = 0 \text{ in } \Omega$$

with $q = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}$. It is easy to see that

$$\Lambda_q(f) = \gamma^{-\frac{1}{2}}|_{\partial\Omega} \Lambda_\gamma(\gamma^{-\frac{1}{2}}|_{\partial\Omega} f) + \frac{1}{2}(\gamma^{-1}\frac{\partial\gamma}{\partial\nu})|_{\partial\Omega} f.$$

Now Kohn and Vogelius showed in [KV] that given any open subset Γ of $\partial\Omega$, if we know $\Lambda_\gamma(f)|_\Gamma$ for all f then we can determine $\gamma|_\Gamma$ and $\frac{\partial\gamma}{\partial\nu}|_\Gamma$, reducing therefore the proof of Corollary 0.2 to Theorem 0.1.

The authors thank G. Dytalov for his careful reading of the manuscript.

References

[DL] Dautray, Lions J.-L. Mathematical Analysis and Numerical Methods for Science and Technology, Vol. 2, Springer-Verlag, Berlin, Heidelberg, 1993.

[GU] Greenleaf G., Uhlmann G., Local uniqueness for the Dirichlet-to-Neumann map via the two plane transform, preprint.

[KV] Kohn R., Vogelius M., Determining conductivity by boundary measurements, Comm. Pure Appl. Math., **37** (1984), 289–298.

[LM] Lions J.-L., Magenes E. Problemes aux Limites Non Homogenes et Applications, Vol. 1, Dunod, Paris, 1970.

[N] Nachman A. I. Global Uniqueness for a two-dimensional inverse boundary value problem, Ann. of Math., **143** (1996), 71–96.

[SU] Sylvester J., Uhlmann G. A global uniqueness theorem for an inverse boundary value problem, Ann. of Math., **125** (1987), 153–169.

[SuU1] Sun Z. Q., Uhlmann G. Generic uniqueness for an inverse boundary value problem, Duke Math. J. **62** (1991), 131–155.

[SuU2] Sun Z. Q., Uhlmann G. Recovery of singularities for formally determined inverse problems Comm. Math. Phys., **153** (1993), 431–445.

[T] Taylor M. Partial Differential Equations Vol. 1, Springer-Verlag, New York, 1996.