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Inverse Problems in Industrial Applications

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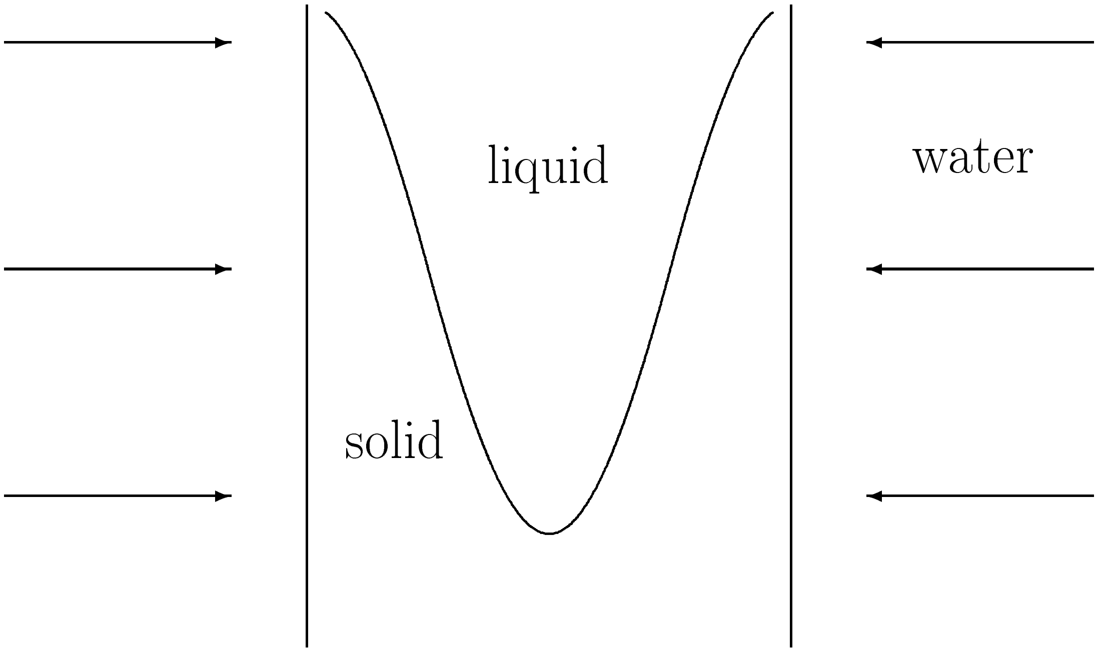
Project Group 1: Continuous Casting / Rolling of Steel (VOEST–Alpine; coworkers: T.Langthaler, P.Manselli, A.Binder, S.Vessella, W.Grever, G.Landl, E.Radmoser)



For metallurgical and technological reasons: phase boundary as function of time / position important!

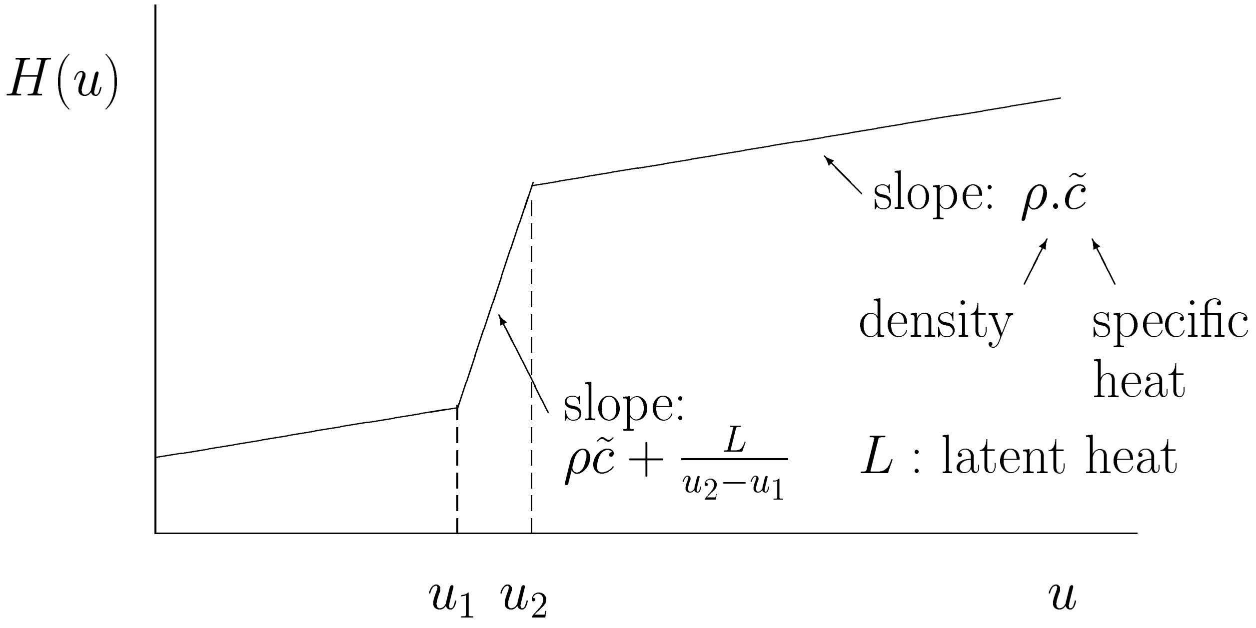
Aim: Set spray cooling in such a way that a given phase boundary ("solidification front") is approximated as closely as **necessary**!

Mathematical model:

Heat conduction in casting direction: negligible (by 10^{-4} smaller), constant density

First model:

$H(u)$: enthalpy at temperature $u = u(x, y, t)$



$$\Rightarrow \text{div}(k(u).\text{grad } u) = \frac{\partial H(u)}{\partial(t)} =: \rho.c(u).\frac{\partial u}{\partial t}$$

$k(u)$ … thermal conductivity

$c(u)$ … "effective heat capacity"

$[u_1, u_2]$ … solidification interval

$$\operatorname{div}(k(u)\operatorname{grad} u) = \rho c(u)\tfrac{\partial u}{\partial t} \quad \text{in strand}$$

$$k(u).u_x = g_v(t)[u - U_w] \quad \text{on boundary in } x\text{–direction}$$

$$u = f \quad \text{initial temperature}$$

$$u(S(t), t) = u_1$$

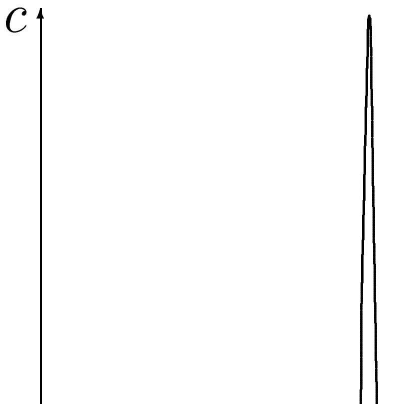
+ radiation + boundary conditions in y–direction

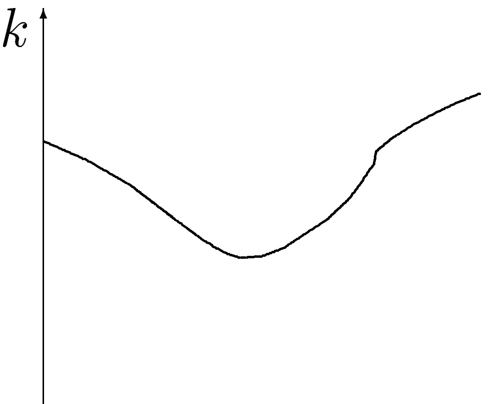
u : temperature

U_w : water temperature

u_1 : temperature of complete solidification

$g_v(t)$: heat transfer function,
depends on casting speed v





Further simplifications:

length $\geq 3*$ width (slab casting): one–dimensional model justified

symmetry

$$\frac{\partial}{\partial x}(k(u)\frac{\partial u}{\partial x}) = \rho c(u)\frac{\partial u}{\partial t} \text{ in strand}$$

$$k(u(0,t)).\frac{\partial u}{\partial x}(0,t) = g_v(t)[u(0,t) - U_w]$$

$$\frac{\partial u}{\partial x} = 0 \text{ in the center of the strand}$$

$$u(x,0) = f(x)$$

$$u(s(t),t) = u_1$$

Direct problem: Given g_v, compute u, s.

Inverse problem: Compute $\mathbf{g_v}$, everything else given (especially s!).

Similar to inverse Stefan problem (ill–posed!), but: solidification not at a fixed temperature, but within an interval of temperatures!

Theoretical results:

a) Engl–Langthaler–Manselli, ISNM 78 (1987):

- uniqueness for the inverse problem
- well–posedness for the direct problem
- continuous dependence for the inverse problem under the assumption of an a–priori bound for $|g_v'|$
  (“regularization by compactness”, no quantitative stability result!)

b) Engl–Manselli, Num.Funct.Anal.Opt. (1989):
Rates for a related (simple) linearized problem (Cauchy problem for heat equation with Cauchy data given up to a specific time):

Find $u(1,.) =: f_0 \in L^2(I\!R)$ from noisy measurements g_δ $(\|g_\delta - g_0\|_{L^2(I\!R^-)} \leq \delta)$ of g_0 :

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial t} \quad 0 < x < 1 \ , t \in I\!R \\ \frac{\partial u}{\partial x}(0,t) &= 0 \qquad t \in I\!R \\ u(0,t) &= g_o(t) \quad t < 0. \end{aligned}$$

Using Fourier and Hilbert transform

$\Rightarrow$ quantitative stability result:

Theorem: $\dot{f} \in L^2(I\!R)$, $\|f\|_{L^2(I\!R)} + \|\dot{f}\|_{L^2(I\!R)} \leq E$; then for any $n > 2$ and $t \leq 0$,

$$|f_0(t) - f_\delta(t)| \leq K_n.E.(\log \frac{E}{\delta})^{-\frac{1}{4}+\eta_n}$$

with $K_n \to \infty, \eta_n \to 0$ as $n \to \infty$.

Very weak stability!

More smoothness for f :

$$|f_0(t) - f_\delta(t)| = o(\sqrt{\delta}),$$

realizable with Tikhonov regularization!

c) Binder–Engl–Vessella (Num.Funct.Anal.Opt. 1990): For the original (nonlinear)problem: Under smoothness assumptions for the coefficient functions:

$$\|g_v - g_v^\delta\|_\infty \leq c[\delta + (\log \frac{1}{\delta})^{-\frac{1}{3}}],$$

if

$$\|s - s_\delta\|_\infty \leq \delta$$

and

$$\|g_v\|_\infty + \|\ddot{g}_v\|_\infty \leq E.$$

Convergence and stability of Tikhonov regularization based on Engl–Kunisch–Neubauer, Inverse Problems 1989 (cf. first talk)

PRACTICAL APPROACH:

Similar work:

Saguez:	control of enthalpy by secondary cooling
Neittaanmäki:	surface temperature and its variation controlled by secondary cooling in casting of billets (2–D)
Australia (UNSW, CSIRO, BHP):	suggestions for an approach similar to ours

Our approach:

$g_v(t)$ can be controlled independently in 6 zones

$\Rightarrow$ regularization by restricting the number of degrees of freedom ($\leftrightarrow$ regularization by compactness)

$z_1, \ldots, z_6$: heat transfer coefficients
in 6 spray cooling zones, determine $g_v(t)$
(together with v)

$s(z, v)$: corresponding solution (solidification front)
for the direct problem

s^*: prescribed solidification front

$$\sum_{i=1}^{n} \gamma_i [s(z,v)(t_i) - s^*(t_i)]^2 \rightarrow \text{ min.}$$

$$a_j \leq z_j \leq b_j, v_{\min} \leq v \leq v_{\max}.$$

Nonlinear optimization problem (6-7 variables) with simple linear constraints,

γ_i : appropriate weights.

Each evaluation of the objective function =
solution of the direct problem!

Method used: Rosen's method (gradient projection), line search

Tests show: influence of casting speed large.

Practice: casting speed should be predetermined
$\rightarrow v_{\min} = v_{\max}$.

Actual examples (VOEST–Alpine AG):

Example 1: $v = 1.5$ m/min, $n = 13$, 3 iterations:

t_i(sec)	γ_i	given solidif.	achieved front (mm)	rel. error (%)
34	20	16.8	16.8	0
42	20	17.5	17.5	0
60	20	21.9	22.1	0.91
86	10	26.7	27.0	1.12
118	10	31.6	31.9	0.95
170	5	37.9	38.8	2.37
240	5	45.8	45.6	-0.44
340	4	55.2	54.5	-1.27
480	4	65.7	65.0	-1.07
670	3	78.7	77.2	-1.91
950	2	94.4	94.7	0.32
1340	1	114.4	115.4	0.87
1540	0.8	129.5	129.5	0

Example 2: $v = 1.9$ m/min, n, γ_i as in Example 1:

t_i(sec)	given solidif.	achieved front (mm)	error (%)
28	16.8	16.4	-2.38
36	17.5	17.1	-2.29
50	21.9	21.5	-1.83
72	26.7	26.4	-1.12
98	31.6	31.1	-1.58
145	37.9	36.9	-2.64
200	45.8	45.2	-1.31
285	55.2	54.7	-0.91
400	65.7	65.7	0
560	78.7	77.9	-1.02
790	94.4	94.4	0
1120	113.4	114.0	0.53
1300	129.5	129.5	0

More recent work:

Two related **nonlinear inverse problems** for

- **continuous casting**
- **hot rolling**

of steel, methodologically different. In each problem, two inverse subproblems of markedly different type:

- control of temperature/phase change via boundary cooling ("cause for a **desired** effect")
- determination of heat transfer functions from temperature measurements ("cause for an **observed** effect"), much more unstable, uniqueness question.

Problem: In the point of complete solidification (“crater end”): Segregation and thermal stresses because of strongly different temperature gradients in the interior and on the boundary of the strand

Aim: To avoid segregation by “soft–reduction” (slightly reducing the diameter of the strand in the soft-reduction-zone)

Our inverse problem:

Desired effect: complete solidification within the soft-reduction zone

“Cause”: Secondary cooling of the strand by cooling water

Main problem: **variable casting speed**

Mathematical Model:

– Lagrangian coordinates

– Heat conduction in casting direction negligible

$\Rightarrow$ spatially two-dimensional problem: for every cross-section z, the temperature $u = u(x, y, z, t)$ fulfills

$$\mathrm{div}_{xy}\,[k(u)\,\mathrm{grad}_{xy}\,u] = \tfrac{\partial}{\partial t}[\rho(u)c(u)u]$$

$$u(x, y, z = -\textstyle\int_0^t v(\tau)\,d\tau, t) = f(x, y, t)$$

$$u_x(d/2, y, z, t) = 0, \quad u_y(x, b/2, z, t) = 0$$

$$\begin{aligned} & k(u(0, y, z, t))\,u_x(0, y, z, t) = \\ & \qquad \sigma\epsilon\,(u(0, y, z, t)^4 - u_a^4) + \\ & \qquad g(z + \textstyle\int_0^t v(\tau)\,d\tau, t) \;\cdot\; (u(0, y, z, t) - u_w) \\ & k(u(x, 0, z, t))\,u_y(x, 0, z, t) = \\ & \qquad \sigma\epsilon\,(u(x, 0, z, t)^4 - u_a^4) + \\ & \qquad g(z + \textstyle\int_0^t v(\tau)\,d\tau, t) \;\cdot\; (u(x, 0, z, t) - u_w) \end{aligned}$$

Heat transfer function g describes cooling by spray water

k : heat conductivity

v : casting speed

u_a : air temperature

u_w : water temperature

Solution of the inverse problem by minimizing the functional

$$\begin{aligned} F(g) \;=\; & \int_{z_1}^{z_2} P\left(\int_{t(z)}^{t(z,g)} v(\tau)\, d\tau - L_l\right) \\ & + P\left(L_u - \int_{t(z)}^{t(z,g)} v(\tau)\, d\tau\right)\, dz \end{aligned}$$

over all admissible heat transfer functions g;

$[L_l, L_u]$: soft reduction zone

penalty function P: differentiable with $P(s) = 0$ for $s \leq 0, P(s) > 0$ for $s > 0$. $t(z, g)$: time when cross section z is completely solid, depends on unknown g!

$\Rightarrow$ Complicated nonlinear optimization problem, unstable.

But: g is piecewise constant (finitely many cooling zones, constant cooling in each; finitely many times for changing cooling)

⇒ built-in regularization!

Extensions: Overheating (material parameters and solidus temperature depend, for each cross-section, on the initial temperature). Mathematical theory for direct problem with piecewise constant boundary functions, efficient numerical solution of inverse problem: W. Grever (now Mannesmann:

- Solution of optimization problem with a Quasi-Newton method
- derivative computed via adjoint problem (decisive for speed, especially for higher dimensions: blooms, billets).

Numerical example: acceleration from 1.6 to 3.2 m/min. in about 10 minutes CPU on DEC 3000/600 AXP:

How to determine heat transfer functions in boundary condition?

Available measurements: Surface temperature at a few points on strand surface.

Project done by C. Carthel:

Traditional approach (literature, VAI):

$g(q)$: heat transfer function as function of amount of water q, represented as

$$g(q) = a \cdot q^b,$$

"determine" a, b by

$$\min_{a,b>0} \|F(a,b)\|^2$$

with

$$F(a,b) = \begin{bmatrix} u_1(a,b) - u_1^* \\ u_2(a,b) - u_2^* \\ \vdots \\ u_N(a,b) - u_N^* \end{bmatrix},$$

$u_i(a,b)$: simulated temperature using

$$g(q) = a \cdot q^b,$$

u_i^* : measured temperature; i counts cross-sections and/or pyrometers.

Yields completely useless results!

The problem is very unstable, combined backwards/sideways heat equation!

A bit, but not much better with Tikhonov regularization

$$\|F(g)\|^2 + \alpha \|g''\|^2 \to \min .$$

Too few degrees of freedom!

Next attempt: subdivide q-interval, represent g as piecewise linear/cubic spline in q.

Problems, since data are very unevenly distributed in q-space (i.e., over water flow rates), more serious for cubic case.

Satisfactory results:

$$\|F(g)\|^2 + \alpha \|g''\|^2 \to \min .$$
$$g \in \{\text{cubic splines}\}$$

M : number of subintervals for spline

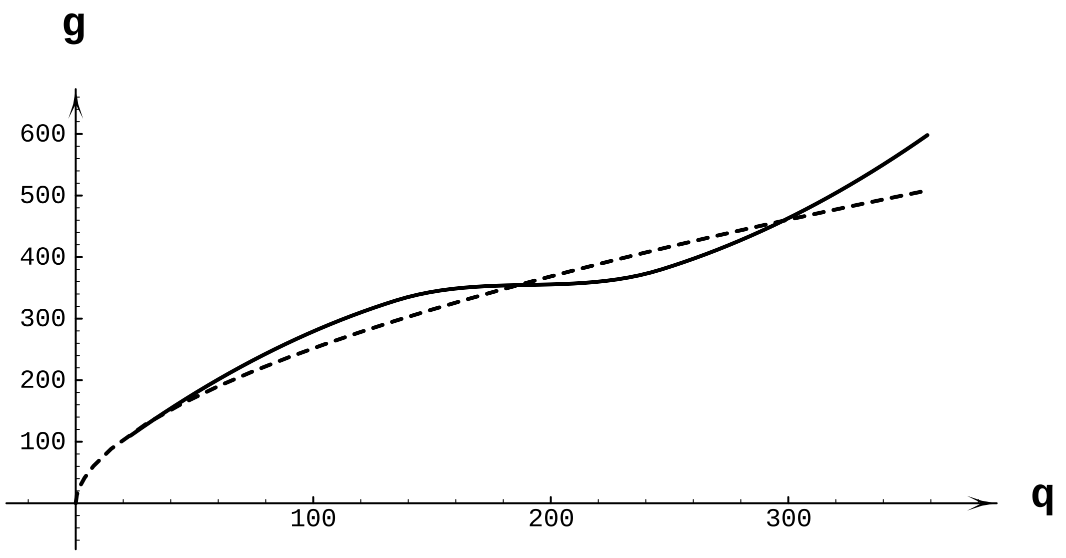
N : number of temperature measurements

Results without artificial noise (but computational noise):

With noise (E: noise in C):

E	$\frac{\|R\|_2}{\|F(0)\|_2}$	**Iterations**	$\frac{\|F(g)\|_2}{\|F(0)\|_2}$	$\frac{\|g-g^*\|_{L^2}}{\|g^*\|_{L^2}}$
5.0	2.21×10^{-2}	11	2.08×10^{-2}	0.0803
10.0	4.76×10^{-2}	11	4.65×10^{-2}	0.264
20.0	8.68×10^{-2}	16	8.21×10^{-2}	0.306

Summary of numerical results using the cubic spline form with simulated data, simulated noise $N = 40$ and $M = 4$.



The piecewise cubic heat transfer function obtained using simulated data with simulated noise $(E = 5)$, $M = 4$, $N = 40$ and $\alpha = 14.1$.

Works reasonably also for actual data (results reproducable for different data sets)!

Similar situation: Cooling of steel after hot rolling

Aim: The coiling temperature should be constant over the whole length of the steel strand, **but the rolling speed increases ("speed–up")**.

The way the strand is cooled down is decisive for steel quality and should be constant over the whole strand (in spite of speed–up) and as close as possible to a prescribed function.

Project performed by G. Landl

There are many cooling devices, they can only be shut on or off, this should not be done too often.

Inverse problem:

Desired effect: The temperature of the steel strand should always and everywhere be as close as possible to a prescribed value

"Cause": Switching rules for the cooling devices during the process

Mathematical model:

– Heat conduction in rolling direction negligible

– Temperature over the cross-section constant

$\Rightarrow$ (spatially) one–dimensional heat equation for every cross section z:

$$\frac{\partial}{\partial x}\left(k(u)\cdot\frac{\partial u}{\partial x}(x,z,t)\right)=\frac{\partial}{\partial t}[\rho(u)\,c(u)\,u]$$

$$u(x,z,\tau_0(z))=f(x,z)$$

$$-k(u(-\tfrac{d}{2},z,t))\cdot\frac{\partial u}{\partial x}(-\tfrac{d}{2},z,t)$$

$$=Q(p_-(\zeta,t)\cdot w_-(\zeta),\ u)$$

$$k(u(\tfrac{d}{2},z,t))\cdot\frac{\partial u}{\partial x}(\tfrac{d}{2},z,t)$$

$$=Q(p_+(\zeta,t)\cdot w_+(\zeta),\ u)$$

$$\zeta=\zeta(z,t):=\int_{\tau_0(z)}^{t}v(\tau)d\tau$$

$$z=\int_0^{\tau_0(z)}v(\tau)d\tau$$

The function p describes the cooling strategy:

$$p_\pm(\zeta,t)=\begin{cases}0 & \text{“water–cooling off”}\\ 1 & \text{“water–cooling on”.}\end{cases}$$

Solution of the inverse problem by minimizing

$$\int\limits_{0}^{\Theta} \int\limits_{\tau_0(z)}^{T(z)} \omega(z,t) \cdot \Big(\tfrac{1}{2\,x_0} \textstyle\int_{-x_0}^{x_0} u(x,z,t)dx$$
$$-\tilde{u}(z,t)\Big)^2 dt\, dz + \alpha\, S(p)$$

over all cooling strategies p via discrete optimization. $S(p)$ is the number of switches (**is a regularization!**).

Additional requirements on cooling strategies (e.g., no switching–off) can be incorporated.

Numerical example:

– Strand with thickness 4 mm

– Speed–up from 6 of 12 m/sec

– 60 cooling devices

– CPU–time on DEC 3000/60 AXP: 17 min.

Also here:
Identification of heat–transfer function from surface temperature measurements (G. Landl, E. Radmoser).

Additional difficulty: in this temperature range, phase change from austenite to perlite takes place (maybe only incompletely), influences temperature measurements.

Approach:

- physically motivated model for g as a function of water flow rate and temperature, several (device–dependent) groups of free parameters
- heat capacity $k(u)$ contains a part describing the phase change: represented as cubic spline with prescribed integral (= known latent heat of phase change if it takes place completely)

→ Optimization problem, several (local–global) regularization terms, proper scaling, minimization by Levenberg-Marquardt.

Reasonable results for real data, outliers for data with incomplete phase change:

Project 2: Nondestructive Testing

(Hilti AG, Liechtenstein; coworker: A.Neubauer)

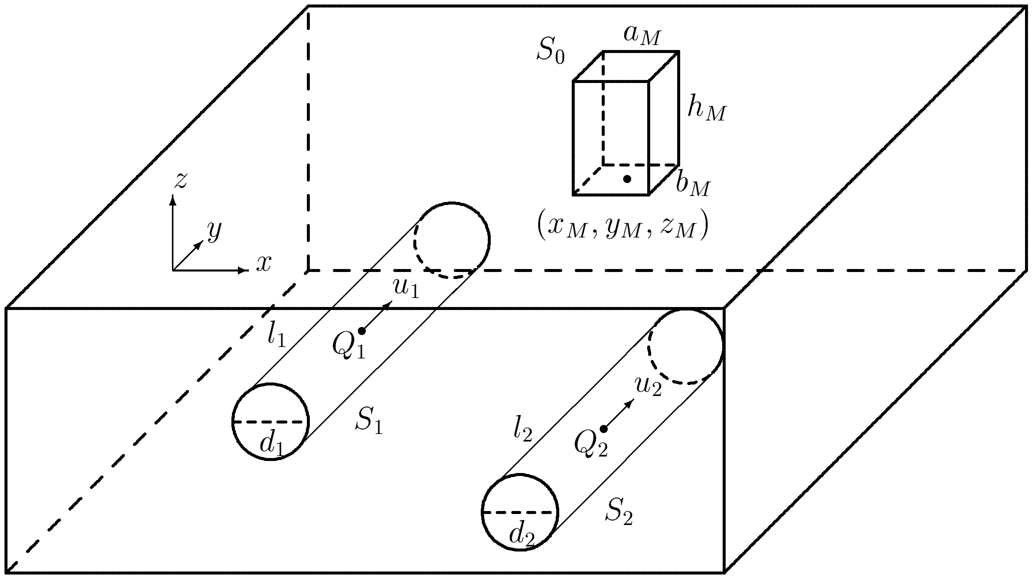
Aim:	identification of steel
	reinforcement bars in concrete
In practice:	various methods available
	inaccurate and/or expensive,
	unfeasible for construction site!

Company wants to use magneto**static** field distortion!

Identifiability (uniqueness question)?

Stability?

Are there reasonably fast algorithms for 3–D problem?



The relevant parameters are:

a_M, b_M, h_M:	dimensions of the permanent magnet
(x_M, y_M, z_M):	coordinates of the center of one face of the magnet
n:	number of bars
d_i, l_i:	diameter and length of S_i
$u_i = (ux_i, uy_i, uz_i)$:	unit vector in direction of the axis of S_i
$Q_i = (Qx_i, Qy_i, Qz_i)$:	center of S_i

Assume: magnetic induction B related to the magnetic field H in a linear way, i.e., $B = \mu H$; μ: magnetic permeability, $\mu = \mu_0 > 0$ in $I\!R^3 \setminus \cup_{i=1}^{n} \bar{S}_i$, $0 < \mu = \mu_i \neq \mu_0$ in $\bar{S}_i$, $i = 1, \ldots, n$.

Permanent magnet S_0: uniformly magnetized in negative z–direction.

Direct Problem: Given the location and the magnetization of S_0 and the locations and shapes of the $S_i (i = 1, \ldots, n)$, compute the total magnetic field $H = H^i + H^s$, where H^i is the "incident field" generated by S_0 if no enclosures $S_i (i = 1, \ldots, n)$ are present; H^s is called the "scattered field".

Inverse Problem: Given the location and the magnetization of S_0 (and hence the incident field H^i), determine the S_i from measurements of certain components of the field H (which give the corresponding components of the scattered field H^s).

Mathematical model for the direct problem:

Maxwell's equations:

$$\begin{aligned}
B =&\ \mu_0(H+M) && \text{in } S_0\\
B =&\ \mu_0 H && \text{in } I\!R^3 \setminus \bigcup_{i=0}^{n} \bar{S}_i\\
B =&\ \mu_1 H && \text{in } \bigcup_{\mathrm{i=1}}^{\mathrm{n}} \bar{\mathrm{S}}_{\mathrm{i}}\\
\text{div B} =&\ 0 && \text{in } I\!R^3 \setminus \bigcup_{i=0}^{n} \partial S_i\\
\text{rot B} =&\ 0 && \text{in } I\!R^3 \setminus \bigcup_{i=0}^{n} \partial S_i\\
(n_i, B)_+ =&\ (n_i, B)_- && \text{on } \partial S_i \text{ for } i \in \{0,\dots,n\}\\
[n_i, H]_+ =&\ [n_i, H]_- && \text{on } \partial S_i \text{ for } i \in \{0,\dots,n\}\\
H(x) =&\ o(1) && \text{as } \parallel x \parallel \to +\infty\\
& && \text{uniformly in all directions.}
\end{aligned}$$

Reduction to a boundary integral equation:

$$\gamma(x,y) := \frac{1}{4\pi \parallel x-y \parallel} \quad (x \neq y \in I\!R^3);$$

Cauchy's integral formula for harmonic vector fields

$\Rightarrow$

$$H(x) = -\mathrm{grad}_x \int_{\partial V} \varphi(y)\gamma(x,y)dS(y) \quad (x \in I\!R^3 \backslash \partial V)$$

with

$$\varphi(y) := (n(y), H(y))_+ - (n(y), H(y))_- \quad (y \in \partial V),$$

$$\partial V := \bigcup_{i=0}^{n} S_i.$$

φ is a solution of

$$\frac{\mu_0+\mu_1}{2(\mu_1-\mu_0)}\varphi(x) + \sum_{i=1}^{n} \int_{\partial S_i} \varphi(y)\frac{\partial}{\partial n(x)}\gamma(x,y)dS(y) =$$

$$= -\int_{\partial S_0} (n(y), M)_- \frac{\partial}{\partial n(x)}\gamma(x,y)dS(y)$$

$$(x \in \partial S_j, j \in \{1,\ldots,n\}).$$

Problems are equivalent $\Rightarrow$ well–posedness for direct problem.

Note: $u := \int_{\partial V} \varphi(y)\gamma(.,y)dS(y)$ solves

$$\left.\begin{array}{lcll} \triangle u & = & 0 & \text{in } I\!R^3 \setminus (\cup_{i=0}^n \partial S_i \cup S_0) \\ u_+ & = & u_- & \text{on } \partial S_i \ (i \in \{0,\dots,n\}) \\ \mu_0(\frac{\partial u}{\partial n})_+ & = & \mu_1(\dfrac{\partial u}{\partial n})_- & \text{on } \partial S_i \ (i \in \{1,\dots,n\}) \end{array}\right\} (*)$$

$$\begin{array}{lcll} \triangle u & = & \text{div M} & \text{in } S_0 \\ (\frac{\partial u}{\partial n})_+ & = & (\dfrac{\partial u}{\partial n})_- - (M,n)_- & \text{on } \partial S_0 \end{array}$$

(+ boundary conditions at infinity).

Weak form of (*):

$$\begin{array}{lcl} \text{div}(\mu \ \text{grad} \ u) & = & 0 \\ \mu & = & \begin{cases} \mu_0 \ \text{in } I\!R^3 \setminus \cup_{i=1}^n S_i \\ \mu_1 \ \text{in } \cup_{i=1}^n S_i \end{cases} \end{array}$$

⇒ Inverse problem closely related to “impedance tomography” (identification from boundary measurements).

Results about identifiability / stability:

Kohn–Vogelius (uniqueness), **Dirichlet–**
Sylvester–Uhlmann (stability), **to–Neumann**
Allessandrini (estimates) **map**
Friedman–Isakov–Powell : **one measurement**

Results suggest identifiability!

Are these results directly relevant for our problem? ⇒

Numerical simulation for direct problem

Galerkin method for solving the boundary integral equations with piecewise $\left\{\begin{smallmatrix}\text{linear}\\\text{constant}\end{smallmatrix}\right\}$ basis functions.

φ has singularities at "corners" (Sloan – Spence), integral equation has to be modified there; treatment of singularities:

a) augment space of basis functions (?)

b) grade mesh, compute all integral as accurately as possible!

Computation of coefficients: 4–fold integrals!
For piecewise constant Ansatz functions: Double integrals computed analytically (symbolic computation), remaining two with Gaußian quadrature.

Numerical results "validated" by comparing with an example with known analytic solution (cylinder–ball).

Numerical results for a typical case:

Incident vs. scattered field:

x	z–component of incident field	z–component of scattered field
0	$0.96312 * 10^{+1}$	$0.12748 * 10^{-1}$
3	$0.62225 * 10^{+1}$	$0.50389 * 10^{-2}$
6	$0.21021 * 10^{+1}$	$0.41310 * 10^{-3}$
9	$0.47721 * 10^{+1}$	$-0.24772 * 10^{-3}$
12	$-0.21581 * 10^{-2}$	$-0.23581 * 10^{-3}$
15	$-0.12025 * 10^{+0}$	$-0.16613 * 10^{-3}$

Thus: method **practically infeasible**, scattered field too weak!

Idea: Remove incident field electronically! Done by fixing sensors to permanent magnet in a symmetric way.

Big difference: Magnet moved, for each position only one measurement (linear combination of scattered field) → Dramatic increase in computing time!

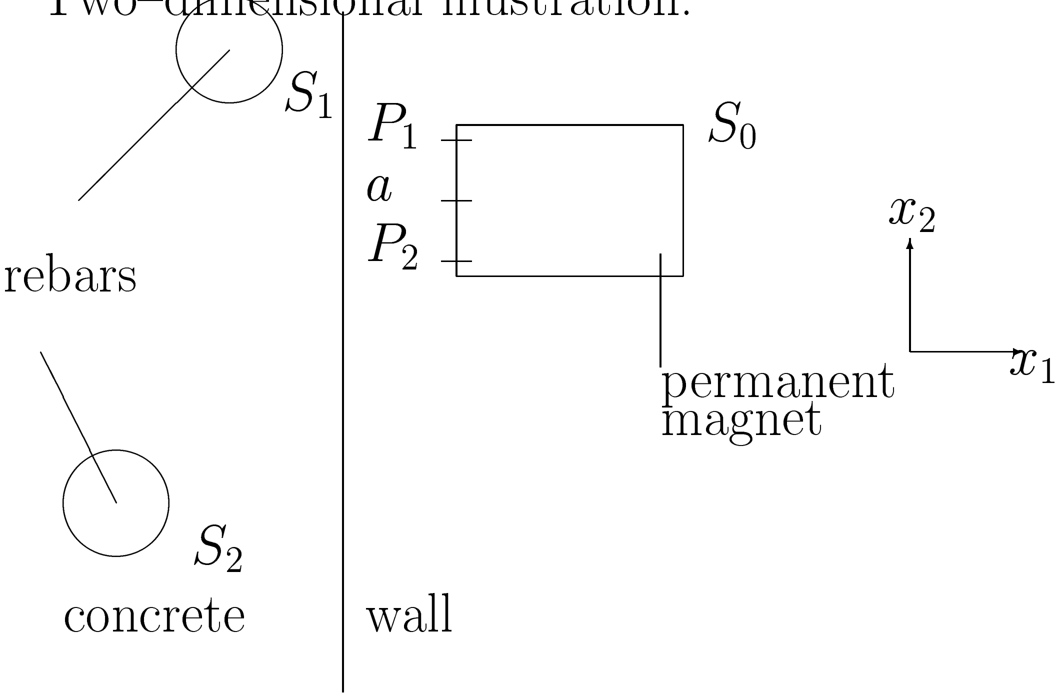
⇒ Automatic solution of inverse problem currently out of reach!

But: use of direct–problem–solver for systematic study of influence of various parameters (location, permeability, depth, diameter of bars) and for optimizing location of sensors:

Simulation replaces experiment!

Lead to prototype with good reconstruction on test examples. **Device on the market!**

Two–dimensional illustration:



Rectangle symbolizes permanent magnet, uniformly magnetized in x_1–direction:

incident field symmetric with respect to a parallel line to the x_1–axis through a. Thus: incident fields at P_1 and P_2 coincide.

u^i:= magnetic potential generated by the magnet (cannot be measured directly): we could e.g. measure $\frac{\partial}{\partial x_1}u^i$, i.e., the component of the magnetic field normal to the wall.

Then:

$$\frac{\partial}{\partial x_1}u^i(P_1) = \frac{\partial}{\partial x_1}u^i(P_2)$$

and also

$$\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1}u^i(a) = 0$$

which essentially corresponds to measuring

$$\frac{\partial}{\partial x_1}u^i(P_1) - \frac{\partial}{\partial x_1}u^i(P_2)$$

for points P_1, P_2 close to and symmetric with respect to a.

Thus,with $u = u^i + u^s$:= total magnetic potential as scattered by the rebars,

$$\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1}u(a) = \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1}u^s(a)$$

In this way, we can actually measure a quantity depending on the scattered field only, which is the desired "subtraction of the incident field" and corresponds to the measurement process mentioned above implemented in the prototype!

The (scalar) magnetic potential u in this example solves (see above, weak form!)

$$\text{div }\left[(1+\mu)\text{ grad } u\right] = \text{div } M \quad \text{in } \mathbb{R}^2$$

with M:= magnetization of the permanent magnet S_0; support of M contained in S_0. Magnetic permeabilities normalized as 1 in air, concrete and S_0, as $1+\mu$ in the rebars.

Now: idealize this problem slightly by considering the permanent magnet as “infinitely small” with its magnetization (in x_1–direction) concentrated at the point a, i.e.,

$$M(y) := (\delta_{y-a}, 0) \quad \text{for } y \in \mathbb{R}^2.$$

Inverse problem of identifying rebars from special measurements described can now be formulated as finding the spatially varying coefficient μ in

$$\text{div } [(1+\mu) \text{ grad } u] = \text{div } M \quad \text{in } \mathbb{R}^2$$

from measurements of

$$\left(\frac{\partial^2}{\partial x_2 \partial x_1} u\right)(a)$$

at sufficiently many points a.

Analysis of underlying mathematical problem (for the three–dimensional case) and proof that μ is identifiable from sufficiently many measurements.

More precisely: for a linearized problem, linearization is justified:
Engl-Isakov (European Journal of Appl. Math. 1992).

Obtain from representation via volume potential:

Inverse Problem equivalent to the following linear integral equation of the first kind for μ:

$$\int_{\Omega} k(a-y)\mu(y)dy = f(a) \quad (a \in B)$$

with data

$$f(a) := \frac{\partial^2}{\partial x_2 \partial x_1} v(.;a)|_{x=a}$$

(v : linearized magnetostatic potential) and kernel

$$k(w) := \frac{3}{16\pi^2} w_2 \cdot \left(\frac{1}{|w|^8} + 4 \frac{w_1^2}{|w|^{10}} \right) \quad (w \in \mathbb{R}^3 \setminus \{0\}).$$

This equation (with some regularization) might give rise to a **direct method** for numerically solving the Inverse Problem, i.e., to a method where one does not have to solve the corresponding direct problem repeatedly.

Numerically too expensive in 3–D?

Theorem: There is at most one solution $\mu \in L^\infty(\Omega)$ to

$$\int_\Omega k(a-y)\mu(y)dy = f(a) \quad (a \in B)$$

and hence to the Inverse Problem.

Proof: Uses Fourier transforms and function-theoretic methods (three complex variables!).

Thus: magnetic permeability distribution μ actually identifiable from the measurements of

$$\frac{\partial^2}{\partial x_2 \partial x_1} v(.;a)|_{x=a}$$

for all $a \in B$. Especially: If cylindrical rebars D have a constant magnetic permeability (which can be, and sometimes is, unknown!) different from that of the surrounding concrete, they can be located. Theoretical foundation (and partial explanation for practical success) of that specific measurement process!

Various other problems like

- optimal shape design for car engine parts
- determination of the thickness of a furnace wall from exterior temperature/heat–flux measurements
- deconvolution of fluourescene data
- design of a 3–D reflector with prescribed illumination distribution (→ Monge-Ampere equation; postdoc Dr. Xu–Jia Wang, Hangzhou)