Exact formulas for a set of orthogonal polynomials

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1 Introduction

This project concerns the computation and use of a non-standard set of orthogonal polynomials. These polynomials have been used by several researchers in the past, but presently not much is known about them analytically. In this project, we will attempt to remedy this situation.

1.1 Orthogonal polynomials

Orthogonal polynomials are useful tools in several numerical methods such as Gaussian quadrature for numerically evaluating integrals, collocation schemes for integro-differential equations and spectral solutions of partial differential equations. A generic set of orthogonal polynomials, which we will denote $Q_l(x)$, satisfies an orthogonality relation with respect to a weight function $w(x)$,

$$\int_a^b w(x)Q_l(x)Q_m(x)dx = M_l\delta_{lm}. \tag{1}$$

Each $Q_l(x)$ is a polynomial in x of order l, that is,

$$Q_l(x) = \sum_{n=0}^{l} c_l^n x^n. \tag{2}$$

Given this fact, if we know $w(x)$, a, b and M_l the c_l^n can be determined uniquely by simply imposing the orthogonality relation between $Q_{l'}(x)$ and

$Q_l(x)$ for all $l' \leq l$. Orthogonal polynomials satisfy a recurrence relation of the general form

$$Q_{n+1}(x) = (\gamma_n x - \alpha_n)Q_n(x) - \beta_n Q_{n-1}(x)\ . \tag{3}$$

Before we introduce the polynomials we will use, we will review two standard sets, the Hermite and associated Laguerre polynomials.

1.2 Hermite polynomials

These are polynomials for which $w(x) = e^{-x^2}$, $a = -\infty$ and $b = \infty$. Also, rather than fixing M_l we may choose to fix one of the c_l^n coefficients, and if we choose $c_l^l = 2^l$ then we get the standard Hermite polynomials and

$$M_l = \sqrt{\pi} w^l l!\ . \tag{4}$$

They appear in many contexts but those who have taken quantum mechanics may recognize them as the wavefunctions of the harmonic oscillator. They are also used in probability theory where the weight function is taken to be $w(x) = e^{-x^2/2}$ and thus the polynomials differ slightly (the so-called probabilists' definition). The first few Hermite polynomials using our definition are

$$\begin{aligned} H_0(x) &= 1 \\ H_1(x) &= 2x \\ H_2(x) &= 4x^2 - 2 \\ H_3(x) &= 8x^3 - 12x. \end{aligned} \tag{5}$$

The n^{th} Hermite polynomial is the solution of the Hermite differential equation,

$$u'' - 2xu' = -2nu \tag{6}$$

which has a the form of a Sturm-Liouville eigenvalue problem. A recurrence relation for the Hermite polynomials is

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x). \tag{7}$$

Comparing this expression with the general form (3) we see that $\alpha_n = 0$, $\beta_n = 2n$ and $\gamma_n = 2$. In practice, recurrence formulas are often used to generate the polynomials.

There are a great many interesting identities for the Hermite polynomials. The Wikipedia page is a good starting point for more information, but the book by Gradshteyn and Ryzhik [1] and the one by Abramowitz and Stegun [2] are indispensible for this project.

1.3 Associated Laguerre Polynomials

The weight $w(x) = e^{-x}x^{\alpha}$ and the domain $[a, b) = [0, \infty)$ leads to the associated Laguerre polynomial of order α, denoted $L_n^{(\alpha)}(x)$. When $\alpha = 0$ we get the ordinary (i.e., not "associated") Laguerre polynomials, $L_n(x)$. The standard normalization of the polynomials leads to

$$M_l^{\alpha} = \frac{\Gamma(l + \alpha + 1)}{l!} \ . \tag{8}$$

The first few of these polynomials are

$$\begin{aligned}
L_0^{(\alpha)}(x) &= 1 \\
L_1^{(\alpha)}(x) &= -x + \alpha + 1 \\
L_2^{(\alpha)}(x) &= \frac{x^2}{2} - (\alpha + 2)x + \frac{(\alpha + 2)(\alpha + 1)}{2} \\
L_3^{(\alpha)}(x) &= \frac{-x^3}{6} + \frac{(\alpha + 3)x^2}{2} - \frac{(\alpha + 2)(\alpha + 3)x}{2} + \frac{(\alpha + 1)(\alpha + 2)(\alpha + 3)}{6} \ .
\end{aligned}$$

The associated Laguerre polynomials are solutions of the Laguerre differential equation,

$$xu'' + (\alpha + 1 - x)u' = -nu \ . \tag{9}$$

As for the Hermite polynomials there are many interesting identities known for the Laguerre polynomials. One example is the sum formula,

$$L_n^{(\alpha+\beta+1)}(x + y) = \sum_{i=0}^{n} L_i^{(\alpha)}(x)L_{n-i}^{(\beta)}(y) \tag{10}$$

which is useful for applications to plasma kinetic theory.

1.4 Our polynomials

The polynomials we plan to study are similar to both the Laguerre and Hermite polynomials. The weight function is the Hermite-like $w(x) = x^{\nu}e^{-x^2}$, but the domain is $[a, b) = [0, \infty)$ as for the Laguerre polynomials. Rather surprisingly, almost nothing is known analytically about these polynomials, which we denote $P_n^{(\nu)}(x)$. Although we can generate them using the (Gram-Schmidt) orthogonalization procedure described in section 1.1, we can't generate them by the recurrence relation because γ_n, α_n and β_n are unknown exactly. We can compute these coefficients individually at the same time as the polynomials but we do not know any closed form expression for them.

We will now give a short example of a calculation to determine the polynomials corresponding to $\nu = 0$ which we will write $P_n(x)$. From equation (2), we can see that

$$P_0(x) = c_0^0 \; . \tag{11}$$

To get the polynomials uniquely, we need to fix a normalization, which we will do here by setting $c_l^l = 1$. Alternatively, we might choose to fix M_l by setting, say, $M_l = 1$. However, by our choice we find $P_0(x) = 1$ and from this we can find $M_0 = \sqrt{\pi}/2$. We can then proceed by writing

$$P_1(x) = x + c_1^0 \; . \tag{12}$$

Enforcing orthogonality with $P_0(x)$ fixes c_1^0 and we find

$$P_1(x) = x - \frac{1}{\sqrt{\pi}} \; . \tag{13}$$

Continuing in this way, we find for the third and fourth polynomials

$$\begin{aligned} P_2(x) &= x^2 - \frac{\sqrt{\pi}}{\pi - 2} x + \frac{4 - \pi}{2(\pi - 2)} \\ P_3(x) &= x^3 - \frac{3\pi - 8}{2\sqrt{\pi}(\pi - 3)} x^2 + \frac{10 - 3\pi}{2(\pi - 3)} x - \frac{16 - 5\pi}{4\sqrt{\pi}(\pi - 3)} \; . \end{aligned} \tag{14}$$

The paper by Landreman and Ernst [3] also gives the first few polynomials for the normalization $M_l = 1$.

2 Project description

Despite their apparent similarity to both the Laguerre and Hermite polynomials, there are no closed form expressions for the polynomials themselves, their recurrence formulas, or even their orthogonality relation. Simply put, we want to rectify this situation if possible. To this end, we have a few specific questions in mind:

1. Do the polynomials $P_n^{(\nu)}(x)$ satisfy a differential equation similar to the Hermite or Laguerre equations? If so, are the polynomials the eigenfunctions of a Sturm-Liouville operator like the other two?

2. What is the closed form for the c_l^n?

3. Setting the normalization $c_l^l = 1$, is there a closed form for the M_l^ν in the orthogonality relation

$$\int_0^\infty x^\nu e^{-x^2} P_l^{(\nu)}(x) P_k^{(\nu)}(x) dx = M_l^\nu \delta_{lk}? \tag{15}$$

   Is this normalization even the best one to use for this?

4. Are there closed form expressions for α_n, β_n and γ_n in the recurrence formula?

Attempting to answer these questions, as well as finding any other identities analogous to what is known for the other polyonmials, will be your task. Keep in mind that this is a research project, meaning that we do not know the answers to these questions or even whether they can be simply answered. We do know that the solutions to these problems, if they exist, are not trivial to find. These polynomials have been independently discovered by at least three different groups over the last fifty years and none of them has given closed-form expressions for these quantities. However, these researchers had applications in mind that did not rely on having such formulas and thus may not have devoted serious effort toward finding them. Even so, the answers we seek must not be totally obvious. As far as we know, the most thorough previous work on these polynomials was done by Landreman and Ernst in 2013 [3] and Shizgal in 1981 [4]. Shizgal also references the work of Copic and Petrisic [5] from the 1960s but this paper is hard to find (Shizgal also mentions that these authors suggested the name "Maxwell polynomials").

These are only the papers we *know* about. Therefore, the first step on this project should be a search of the literature to see if anybody has attempted this already. We think not, but a thorough search should still be done. Probably the best starting point is to look up Shizgal's paper in the Web of Science and Google Scholar and see if any of the articles that cite it are relevant. Also, please see if the UCLA library has the Copic and Petrisic paper or if they can get it from somewhere.

Beyond that, the way you tackle the problem is up to you. By the end of the project we would like to have the following things:

1. A code (Mathematica preferably but Matlab or any other programming language are fine too) that can generate $P_n^{(\nu)}(x)$ for a given n and ν. This can be done even in the absence of exact formulas. This code will also help with the rest of the project.

2. Closed-form expressions for the polynomial's basic properties, as described above. Ideally, we would like to have an identity corresponding to every one that is known for the Hermite and Laguerre polynomials.

3. Last year's RIPS team worked on a project in which the associated Laguerre polynomials were used to solve an integro-differential equation. We would like to see the analysis in Chapter 3, the computation of initial conditions, of their final report repeated using the new polynomials. Once again, this can be done even before part 2 is completed.

If these tasks are successfully completed before the end of RIPS, we will work on redoing other parts of last year's project with the new polynomials.

3 References

[1] Gradshteyn I S and Ryzhik I M *Table of Integrals, Series, and Products* (Academic Press)

[2] Abramowitz M and Stegun I *Handbook of Mathematical Functions* URL `http://people.math.sfu.ca/~cbm/aands/toc.htm`

[3] Landreman M and Ernst D 2013 *J. Comp. Phys.* **243** 130

[4] Shizgal B 1981 *J. Comp. Phys.* **41** 309

[5] Copic M and Petrisic M 1965 *Nuklearni Institut Jozef Stefan Report* **R-465**