Non-equilibrium Dynamics of Second Order Macroscopic Traffic Models

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Introduction

It has been shown through experiments that stop-and-go traffic waves can be dampened via autonomous vehicles (AVs) with ad-hoc controls. So it is important to have predictive traffic models that capture the key phenomenon of phantom traffic jams. Macroscopic models are of particular interest because they look at large segments of roads, which gives insight into where to activate AVs to dampen undesired waves. Moreover, these models are able to capture aggregate effects of AVs without having to resolve the details on the level of vehicles.

Macroscopic models track aggregate quantities. Advantages:

Mathematical: Other types of models converge to them in certain limits
Practical: Best suited for state estimation from sparse GPS data
Computational: Easy to upscale millions of vehicles in simulations
Societal: Desirable for privacy and data security, because they don't resolve individual vehicles

First order macroscopic models

Macroscopic models give a description of the time evolution of the vehicle density $\rho(x,t)$ on a road. ρ evolves according to the continuity equation:

$$\rho_t + (\rho u)_x = 0 \tag{1}$$

where u is the traffic velocity. First order models (e.g. LWR) assume that u is a decreasing function of ρ, i.e. $u = U(\rho)$. This leads (1) to be a scalar hyperbolic conservation law model with flux function $Q = \rho U(\rho)$

$$\rho_t + (Q(\rho))_x = 0 \tag{2}$$

LWR models have two main *shortcomings*: **1.** The real relationship between Q & ρ is not functional. **2.** They satisfy a maximum principle: $\rho(x,t) \leq \max \rho_0(x)$ $\forall x, t$. This is not true: traffic jams can arise even if the initial density was constant.

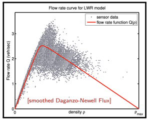


Figure 1: Sensor data showing that the relationship between Q and ρ is not a functional one.

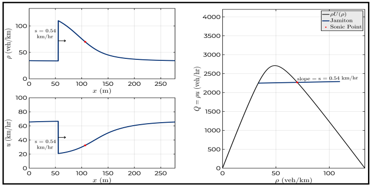
Second order macroscopic models

Second order models augment (1) by an equation for u. One example is the inhomogeneous ARZ model:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (u + h(\rho))_t + u(u + h(\rho))_x = \frac{1}{\tau}(U(\rho) - u) \end{cases}$$

$U(\rho)$ is called the *desired velocity*, $h(\rho)$ is called the *hesitation function*, and τ is a relaxation time that determines how fast drivers adjust to their desired velocity. The second equation is a momentum equation; cars adjust their velocity based on the gap between their velocity u and their desired velocity $U(\rho)$.

Travelling wave solutions to the ARZ



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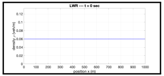
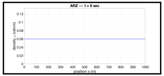
LWR vs ARZ model

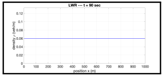
To highlight the major difference between an LWR model & an ARZ model, we run numerical simulations starting with the same constant $\rho_0(x)$ (plus a tiny perturbation) initial conditions. Notice that the LWR $\rho(x,t)$ converged to a constant, while the ARZ $\rho(x,t)$ develops into jams.



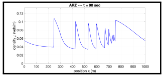
LWR at $t = 0$



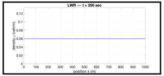
ARZ at $t = 0$



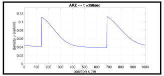
LWR at $t = 90$ sec



ARZ at $t = 90$ sec



LWR at $t = 250$ sec



ARZ at $t = 250$ sec

Jamiton features

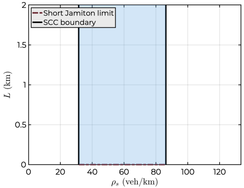
The travelling wave solutions to the ARZ model have several characteristics, some of which are:

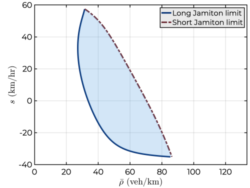
- wave speed: s
- jamiton length: L
- sonic density: ρ_S
- average density: $\bar{\rho}$
- average flow rate: $\bar{Q}$
- etc...

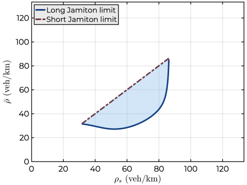
There are infinitely many jamitons that are solutions of the ARZ model. In fact, there are jamitons that are infinitesimally short, and jamitons that are infinitely long, and there are infinitely many jamitons with the same length. To uniquely define a jamiton, at least two features need to be specified.

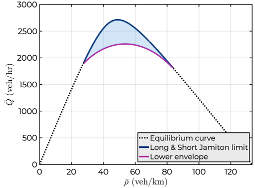
Jamiton spaces

Below are regions of plausible jamitons in four 2-dimensional feature spaces









Research Question: Which Jamitons are stable? Start with Jamiton initial conditions, add a perturbation: What happens to the solution as $t \to \infty$?

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Computational study of jamiton stability

The ARZ model can be solved numerically using a finite volume scheme with an approximate Riemann solver. Use HLL scheme (1st order method).
To study stability, start with a Jamiton solution as initial conditions, add a smooth perturbation at every time step according to the noise equation

$$p^n(x) = \sqrt{\Delta t}\, c(t) \frac{1}{\sqrt{\ell}} \sum_{\nu=1}^{\ell} \xi_\nu^n \sin\left(\frac{2\pi\nu x}{L_0}\right),$$

$$\text{where} \quad \xi_\nu^n \sim \mathcal{N}(0,1)$$

Theoretical analysis of jamiton stability

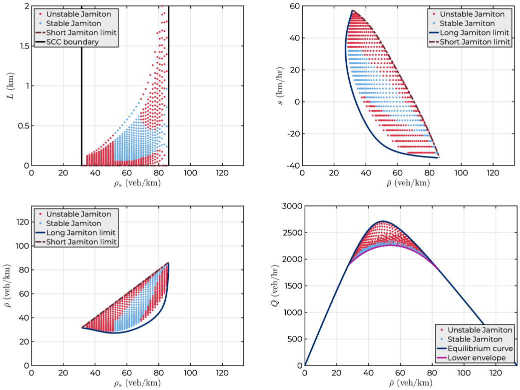
The stability of jamitons is analyzed theoretically by deriving a linear model for small perturbations of a jamiton solution. This analysis is easier in Lagrangian variables, where a perturbation is added to both state variables and the shock position. The resulting linear model can be written as follows:

$$\begin{cases} \delta u_t + b_1(\chi)\delta u_\chi & = a_{11}(\chi)\delta u + a_{12}(\chi)\delta q \\ \delta q_t + m_0 \delta q_\chi & = a_{21}(\chi)\delta u + a_{22}(\chi)\delta q \end{cases}$$

With Robin B.C.: $\delta q(0)_\chi + k_L \delta q(0) = \delta q(L)_\chi + k_R \delta q(L) + c_L \delta u(0) + c_R \delta u(L)$
The coefficient functions are jamiton specific and satisfy the general properties: $m_0 > 0$, $b_1(\chi) = 0$ and $a_{11}(\chi) = 0$ at the sonic pt. A careful high frequency WKB analysis, as well as numerical schemes to solve the linear perturbation model can be found here: `https://arxiv.org/abs/1912.04416`

Jamiton stability results

Running forward simulation for 980 Jamiton profiles, using an HPC cluster.



Region of stability separated by two instability regions: one corresponds to merging (short jamitons combine), and one to splitting (long jamitons split)