I. Setup: CPD and Jennrich’s Algorithm

1. Decompose signal into canonical components.

A tensor \( T \) is a multiindexed array.

\[
T = \sum_{k=1}^{N_t} \mathbf{t}_k \otimes \mathbf{b}_k \otimes \mathbf{a}_k
\]

Canonical Polyadic Decomposition expresses \( T \) as minimal sum of rank 1 terms.

\[
T = \sum_{r=1}^{R} \mathbf{r}_r \otimes \mathbf{v}_r \otimes \mathbf{w}_r
\]

\( R \) is the rank of \( T \).

\( \mathbf{a}_k, \mathbf{b}_k, \mathbf{c}_k \) are vectors of length \( N_x \times N_y \) respectively.

Superscript simplifying assumptions: \( N_x = N_y = N \); i.e. assume \( T \) has low rank.

2. Jennrich’s Eigenvector decomposition gives CPD.

Key idea: Columns of \( T \) are equal eigenvectors of \((T_k^T T_k)\) which in turn are equal to generalized eigenvectors of the matrix pencil \((T_k^T T_k)\) leading to CPD of \( T \).

Notation: \( T_k \) is the \( R \times N \) matrix \( (T_k^T T_k) \).

3. Small eigenvalue gaps lead to instability.

Gen. eigenvalues of \((T_k^T T_k)\) are interpreted as points on the unit circle. The pencil \((T_k^T T_k)\) has \( R \) generalized eigenvalues. Small gaps between gen. eigenvalues causes instability in computing gen. eigenvectors. Instability of Jennrich’s algorithm as \( R \) grows.

Illustration of generalized eigenvalues of \((T_k^T T_k)\). The small gap between generalized eigenvalues \( 1 \) and \( 2 \) is a fundamental source of instability in Jennrich’s algorithm. This effect is quantified by Allman, Breiding, and Vannelli. Jennrich’s algorithm computes the inverse of the matrix of eigenvectors of a pencil \((T_k^T T_k)\). Inverse computation can be avoided by considering “joint generalized eigenvalues” instead of eigenvectors.

3.1 CPD is Jennrich’s algorithm which selects a matrix subpencil of \((T_k^T T_k)\).

Jennrich’s algorithm computes the inverse of the matrix of eigenvectors of a pencil \((T_k^T T_k)\). Inverse computation can be avoided by considering “joint generalized eigenvalues” instead of eigenvectors.

3.2 QZ CPD method: Avoiding inverses.

Jennrich’s algorithm computes an unnecessary inverse.

3.3 Upper triangular slices leads to triangular factors.

3.4 GESD recursively deflates tensor rank.

In practice, GESD recursively writes \( T \) as a sum of tensors of reduced rank.

In our example, GESD uses \( E_1, E_2, E_3, E_4 \) to write the rank 10 tensor \( T \) as

\[
T = T_1^+ + T_2^+ + T_3^+ + T_4^+ + T^-
\]

where \( T_1^+, T_2^+, T_3^+, T_4^+ \) and \( T^- \) have ranks 2, 3, 1, and 4, respectively. \( T^- \) can then be decomposed into a sum of rank 1 tensors using the pencil \((T_k^T T_k)\). Variations in GESD are possible. E.g. one could compute factorizations of eigenvalues as described above rather than working recursively.

In fact, using a single pencil to compute a CPD is a fundamental source of instability in Jennrich’s algorithm. This effect is quantified by Allman, Breiding, and Vannelli. Jennrich’s algorithm computes the inverse of the matrix of eigenvectors of a pencil \((T_k^T T_k)\). Inverse computation can be avoided by considering “joint generalized eigenvalues” instead of eigenvectors.

4. Numerical results

IV. Performance of methods for various tensor ranks.

In Figure 1 indicates a direct improvement on Jennrich’s algorithm (as implemented in Tensorlab’s rank-1 pencil). GESD is the most accurate but slowest method. Multi QZ is the fastest but least accurate method.