Reverse Hölder, Minkowski, And Hanner Inequalities For Complex Matrices

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Abstract

It has been of great interest in recent years to extend known L^p inequalities to general complex matrices under the p-norms $||X||_p = \mathrm{Tr}[(X^*X)^{p/2}]^{1/p}$, where $p \geq 1$. It is well known that Holder and Minkowski inequalities $||X + Y||_p \leq ||X||_p + ||Y||_p$ and $||XY||_1 \leq ||X||_p||Y||_q$ (for conjugate indices p, q) hold. However, we can also define $||X||_s = \mathrm{Tr}[(X^*X)^{s/2}]^{1/s}$ for $s < 1$. For functions, it is known $||fg||_1 \geq ||f||_s||g||_{s/(s-1)}$ provided $g(x) = 0$ on a set of measure 0, and that $||f + g||_s \geq ||f||_s + ||g||_s$ when $f, g \geq 0$. We establish the previously unstudied matrix case for these Reverse Hölder and Reverse Minkowski inequalities. Finally, we comment on how Hanner's Inequality comparing $(||f + g||_p^p + ||f - g||_p^p)$ to $((||f||_p + ||g||_p)^p + |||f||_p - ||g||_p|^p)$ can be extended both to functions and matrices for the $s < 1$ range, and certain related matrix singular value rearrangement inequalities that were previously only known in the $1 \leq p \leq 2$ range that can both be extended to the $2 \leq p \leq 3$ range and the $-1 < s < 1$ range. This combines new research with research in the recent paper https://arxiv.org/abs/2009.04032.

Majorization

We use the technique of majorization for many of the proofs. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ with components labeled in descending order $a_1 \geq \cdots \geq a_n$ and $b_1 \geq \cdots \geq b_n$. Then $\mathbf{b}$ weakly majorizes $\mathbf{a}$, written $\mathbf{a} \prec_w \mathbf{b}$, when

$$\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i, \qquad 1 \leq k \leq n \tag{1}$$

and majorizes $\mathbf{a} \prec \mathbf{b}$ when the final inequality is an equality.

Lemma 0.1. *(Hardy, Littlewood, and Pólya [1] [2]; Tomić, Weyl [6] [7]) Suppose $\mathbf{a} \prec_w \mathbf{b}$. Then for any function $\phi : \mathbb{R} \to \mathbb{R}$ that is increasing and convex on the domain containing all elements of $\mathbf{a}$ and $\mathbf{b}$,*

$$\sum_{i=1}^{n} \phi(a_i) \leq \sum_{i=1}^{n} \phi(b_i). \tag{2}$$

If $\mathbf{a} \prec \mathbf{b}$, the 'increasing' requirement can be dropped.

Lemma 0.2. *Schur [5]; Mirsky [4] Let $X \in M_{n\times n}(\mathbb{C})$ be a self-adjoint matrix with diagonal elements $\mathbf{x} := (x_{11}, \ldots, x_{nn})$. Then*

$$\mathbf{x} \prec \lambda(X) \tag{3}$$

Lemma 0.3. *(Horn [3]) Let $A, B \in M_{n\times n}(\mathbb{C})$. Then*

$$\sigma_\uparrow(A)\sigma_\downarrow(B) \prec_w \sigma(AB) \prec_w \sigma_\uparrow(A)\sigma_\uparrow(B) \tag{4}$$

Reverse Hölder

Theorem 1.1. *Let $A, B \in M_{n\times n}(\mathbb{C})$ with B invertible, and $0 < s < 1$, and $r = \frac{s}{s-1}$. Then*

$$||AB||_1 \geq ||A||_s||B||_r \tag{5}$$

Proof. We directly calculate

$$\begin{aligned} ||AB||_1 &= \sum_{i=1}^{n} \sigma_i(AB) && (6)\\ &\geq \sum_{i=1}^{n} \sigma_i(A)\sigma_{n+1-i}(B) && (7)\\ &\geq \left(\sum_{i=1}^{n} \sigma_i(A)^s\right)^{\frac{1}{s}} \left(\sum_{i=1}^{n} \sigma_{n+1-i}(B)^r\right)^{\frac{1}{r}} && (8)\\ &= ||A||_s||B||_r. && (9)\end{aligned}$$

□

Dual Representation

Theorem 2.1. *Let $A > 0$, and $s < 1$ with r conjugate to s. Then*

$$||A||_s = \inf_{B>0, ||B||_r=1} \mathrm{Tr}[AB], \tag{10}$$

Proof. For all $B > 0, ||B||_r = 1$, we have

$$\mathrm{Tr}[AB] \geq ||A||_s||B||_r = ||A||_s. \tag{11}$$

The infimum is reached by

$$B = ||A||_s^{1-s}A^{s-1}, \tag{12}$$

giving $\mathrm{Tr}[AB] = ||A||_s$. □

Reverse Hanner

Theorem 4.1. *Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then*

$$||\mathbf{x} + \mathbf{y}||_s^s + ||\mathbf{x} - \mathbf{y}||_s^s \leq (||\mathbf{x}||_s + ||\mathbf{y}||_s)^s + \big|||\mathbf{x}||_s - ||\mathbf{y}||_s\big|^s, \tag{13}$$

for $0 < s < 1$, with the inequality reversing when $s < 0$.

Theorem 4.2. *Let $C, D \in M_{n\times n}(\mathbb{C})$ with $C + D, C - D \geq 0$. Then*

$$||C + D||_s^s + ||C - D||_s^s. \leq (||C||_s + ||D||_s)^s + (||C||_s - ||D||_s)^s \tag{14}$$

for $0 < s < 1$, with the inequality reversing when $s < 0$.

We sketch the main proof concepts for Theorem 4.1 and 4.2:

Proof. Assume $||\mathbf{x}||_s \geq ||\mathbf{y}||_s$. Then we can expand the Taylor series

$$(||\mathbf{x}||_s + r||\mathbf{y}||_s)^s + \big|||\mathbf{x}||_s - r||\mathbf{y}||_s\big|^s = 2\sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k)!}||\mathbf{y}||_s^{2k}||\mathbf{x}||_s^{s-2k}r^{2k}. \tag{15}$$

We compare partial sums of this series to the derivatives of

$$F(r) = ||\mathbf{x} + r\mathbf{y}||_s^s + ||\mathbf{x} - r\mathbf{y}||_s^s = \sum_{i=1}^{n} |x_i + ry_i|^s + |x_i - ry_i|^s. \tag{16}$$

to find $F(r)\big|_{r=1} \leq S_{2k}(r)\big|_{r=1}$ for all k when $0 < s < 1$, reversing when $s < 0$.

We prove Theorem 4.2 the $0 < s < 1$ case. The function

$$(x + y)^s + |x - y|^s \tag{17}$$

is strictly decreasing in x for fixed y when $x \leq y$, and strictly increasing in x when $x > y$.

We consider $C + D$, $C - D$ in the basis where C is diagonal. Then as $C + D_{\mathrm{Diag}}, C - D_{\mathrm{Diag}} \geq 0$, we have $||C||_s \geq ||D_{\mathrm{Diag}}||_s \geq ||D||_s$. Treating $||D_{\mathrm{Diag}}||_s$ and the x of Equation (17), and applying first majorization then Theorem 4.1

$$\begin{aligned} ||C + D||_s^s + ||C - D||_s^s &\leq ||C + D_{\mathrm{Diag}}||_s^s + ||C - D_{\mathrm{Diag}}||_s^s && (18)\\ &\leq (||C||_s + ||D_{\mathrm{Diag}}||_s)^s + (||C||_s - ||D_{\mathrm{Diag}}||_s)^s && (19)\\ &\leq (||C||_s + ||D||_s)^s + (||C||_s - ||D||_s)^s && (20)\end{aligned}$$

The argument reverses for $s < 0$. □

Singular Value Rearrangement

Theorem 5.1. *Let $C, D \in M_{n\times n}(\mathbb{C})$ with $C \geq |D| \geq 0$ and $\sigma_n(C) \geq \sigma_1(D)$. Then*

$$||C + D||_s^s + ||C - D||_s^s \leq ||\sigma_\uparrow(C) + \sigma_\downarrow(D)||_s^s + ||\sigma_\uparrow(C) - \sigma_\downarrow(D)||_s^s \tag{21}$$

for $-1 < s < 0$ and $1 \leq s \leq 2$, with the inequality reversing for $0 < s < 1$ and $2 < s < 3$.

Theorem 5.2. *Let $C, D \in M_{n\times n}(\mathbb{C})$ with $C \geq D \geq 0$. Then*

$$||C + D||_s^s + ||C - D||_s^s \geq ||\sigma_\uparrow(C) + \sigma_\uparrow(D)||_s^s + ||\sigma_\uparrow(C) - \sigma_\uparrow(D)||_s^s \tag{22}$$

for $-1 < s < 0$ and $1 \leq s \leq 2$, with the inequality reversing for $0 < s < 1$ and $2 < s < 3$.

For each of the theorems, the conditions on C and D are necessary.

We sketch the proofs of Theorems 5.1 and 5.2:

Proof. A positive matrix X has the following integral representations, with c being a positive normalization constant depending on s or p:

$$X^s = c_s \int_0^\infty t^s\left(\frac{1}{t + X}\right)dt \qquad X^s = c_s \int_0^\infty t^s\left(\frac{1}{t} - \frac{1}{t + X}\right)dt \tag{23}$$

for $-1 < s < 0$ and $0 < s < 1$ respectively, and

$$X^p = c_p \int_0^\infty \left(\frac{X}{t^2} + \frac{1}{t + X} - \frac{1}{t}\right)t^p dt \qquad X^p = c_p \int_0^\infty \left(\frac{X^2}{t^3} - \frac{X}{t^2} + \frac{1}{t} - \frac{1}{t + X}\right)t^{p+1}dt \tag{24}$$

for $1 < p < 2$ and $2 < p < 3$ respectively. We recover $||X||_s^s$ or $||X||_p^p$ by taking the trace.

We let $H = C + t$, $K = H^{-1/2}DH^{-1/2}$. Then

$$\frac{1}{t + C + D} + \frac{1}{t + C - D} = H^{-1/2}(I + K)^{-1}H^{-1/2} + H^{-1/2}(I - K)^{-1}H^{-1/2} = H^{-1/2}\left(\sum_{k=0}^{\infty} K^{2k}r^{2k}\right)H^{-1/2} \tag{25}$$

We now can apply majorization relationships to Line (25) to see that

$$\frac{1}{t + \sigma_\uparrow(C) + \sigma_\uparrow(D)} + \frac{1}{t + \sigma_\uparrow(C) - \sigma_\uparrow(D)} \leq \frac{1}{t + C + D} + \frac{1}{t + C - D} \leq \frac{1}{t + \sigma_\uparrow(C) + \sigma_\downarrow(D)} + \frac{1}{t + \sigma_\uparrow(C) - \sigma_\downarrow(D)} \tag{26}$$

Using the integral representations of the previous slide, we establish the desired inequalities. □

Reverse Minkowski

Theorem 3.1. *Let $A, B > 0$, and $s < 1$. Then*

$$||A + B||_s \geq ||A||_s + ||B||_s \tag{27}$$

Proof. We use the dual representation to directly calculate

$$\begin{aligned} ||A + B||_s &= \mathrm{Tr}[(A + B)Y] && (28)\\ &= \mathrm{Tr}[AY] + \mathrm{Tr}[BY] && (29)\\ &\geq ||A||_s||Y||_r + ||B||_s||Y||_r && (30)\\ &= ||A||_s + ||B||_s. && (31)\end{aligned}$$

□

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