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Revisiting the equality conditions of the data processing inequality for the sandwiched Rényi divergence

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Abstract

We provide a transparent, simple and unified treatment of recent results on the equality conditions for the data processing inequality (DPI) of the sandwiched quantum Rényi divergence, including the statement that equality in the data processing implies recoverability via the Petz recovery map for the full range of α recently proven by Jenčová [1,2]. We also obtain a new set of equality conditions, generalising a previous result by Leditzky et al [3].

Data processing inequality and its saturation

The Umegaki relative entropy satisfies the data processing inequality under the action of a CPTP map Λ :

$$D(\Lambda(\rho)\,||\,\Lambda(\sigma)) \leq D(\rho\,||\,\sigma)$$

When $\Lambda = \mathrm{Tr}_B$ w.l.o.g., the saturation of DPI is equivalent to the following algebraic equality [4]:

$$\sigma_{AB}^{\beta}\rho_{AB}^{-\beta} = \sigma_A^{\beta}\rho_A^{-\beta}, \quad \forall \beta \in \mathbb{C},$$

which is equivalent to the Petz recovery statement:

$$\exists \mathscr{R}_{\sigma,\mathrm{Tr}_B}, \text{ such that } \mathscr{R}^{♤}_{\sigma,\mathrm{Tr}_B}(\rho_A) = \rho_{AB}, \mathscr{R}_{\sigma,\mathrm{Tr}_B}(\sigma_A) = \sigma_{AB}$$

The same DPI and equality condition holds also for the Petz Rényi divergence:

$$\overline{D}_\alpha(\rho\,||\,\sigma) := \frac{1}{\alpha}\log\langle\rho^{\frac{1}{2}}\,|\,\Delta_{\sigma,\rho}^{-\alpha}\,|\,\rho^{\frac{1}{2}}\rangle, \quad \alpha \in (-1,0) \cup (0,1)$$

where $\Delta_{\sigma,\rho}(\,\cdot\,) := \sigma \cdot \rho^{-1}$ is the relative modular operator.

The $\alpha \to 0$ limit gives the Umegaki relative entropy.

We study here the saturation of DPI for the **sandwiched Rényi divergence**, which can also be defined in terms of the relative modular operator [5].

The sandwiched Rényi Divergence $\widetilde{D}_\alpha$

$$\widetilde{D}_\alpha(\rho\,||\,\sigma) := \frac{1}{\alpha}\inf_\omega \log\langle\rho^{\frac{1}{2}}\,|\,\Delta_{\sigma,\omega}^{-\alpha}\,|\,\rho^{\frac{1}{2}}\rangle, \quad \alpha \in (-1,0), \qquad \widetilde{D}_\alpha(\rho\,||\,\sigma) := \frac{1}{\alpha}\sup_\omega \log\langle\rho^{\frac{1}{2}}\,|\,\Delta_{\sigma,\omega}^{-\alpha}\,|\,\rho^{\frac{1}{2}}\rangle, \quad \alpha \in (0,1)$$

The optimiser can be constructed explicitly: $\omega^* \propto \left(\rho^{\frac{1}{2}}\sigma^{-\alpha}\rho^{\frac{1}{2}}\right)^{\frac{1}{1-\alpha}} = \rho^{\frac{1}{2}}\left(\rho^{-1}\overset{♧}{\sharp}_{\frac{1}{1-\alpha}}\sigma^{-\alpha}\right)\rho^{\frac{1}{2}}$

Consider the DPI saturation : $\widetilde{D}_\alpha(\rho_A\,||\,\sigma_A) = \widetilde{D}_\alpha(\rho_{AB}\,||\,\sigma_{AB})$ and let $a_*^2 := \rho_A^{-1}\sharp_{\frac{1}{1-\alpha}}\sigma_A^{-\alpha}$, $\quad b_*^2 := \rho_{AB}^{-1}\sharp_{\frac{1}{1-\alpha}}\sigma_{AB}^{-\alpha}$

The proof of the DPI can be summarised in one line, where the inequality (1) follows from Jensen's operator inequality and inequality (2) follows from the variational definition of $\widetilde{D}_\alpha$.

$$\langle\rho_A^{\frac{1}{2}}\,|\,\Delta^{-\alpha}_{\sigma_A,\rho_A^{1/2}a_*^2\rho_A^{1/2}}\,|\,\rho_A^{\frac{1}{2}}\rangle \leq \langle\rho_{AB}^{\frac{1}{2}}\,|\,\Delta^{-\alpha}_{\sigma_{AB},\rho_{AB}^{1/2}(a_*^2\otimes\mathbb{I}_B)\rho_{AB}^{1/2}}\,|\,\rho_{AB}^{\frac{1}{2}}\rangle \leq \langle\rho_{AB}^{\frac{1}{2}}\,|\,\Delta^{-\alpha}_{\sigma_{AB},\rho_{AB}^{1/2}b_*^2\rho_{AB}^{1/2}}\,|\,\rho_{AB}^{\frac{1}{2}}\rangle$$

Now set the inequalities to equalities

Results

Equality condition (1) (Jenčová) [1,2]

$$\mathscr{R}_{\sigma,\mathrm{Tr}_B}(\rho_A) = \rho_{AB}, \mathscr{R}_{\sigma,\mathrm{Tr}_B}(\sigma_A) = \sigma_{AB}$$

Equality condition (2) (Leditzky, Rouzé and Datta) [3]

$$\rho_A\sharp_{\frac{1}{1-\alpha}}\sigma_A^{\alpha} \otimes \mathbb{I}_B = \rho_{AB}\sharp_{\frac{1}{1-\alpha}}\sigma_{AB}^{\alpha}\,.$$

New equality condition from (1)+(2)

$$\sigma_A^{\beta}(\rho_A\sigma_A^{-\alpha})^{\frac{\beta}{\alpha-1}} \otimes \mathbb{I}_B = \sigma_{AB}^{\beta}(\rho_{AB}\sigma_{AB}^{-\alpha})^{\frac{\beta}{\alpha-1}}, \quad \forall\beta \in \mathbb{C}\,.$$

$\beta = -\alpha$

Footnotes

♧ $\sharp_\lambda$ denotes the geometric mean weighted by λ :

$$A\sharp_\lambda B := A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\lambda A^{\frac{1}{2}}\,.$$

♤ The Petz recovery map:

$$\mathscr{R}_{\sigma,\Lambda}(\,\cdot\,) := \sigma^{\frac{1}{2}}\Lambda^*\left(\Lambda(\sigma)^{-\frac{1}{2}}\cdot\Lambda(\sigma)^{-\frac{1}{2}}\right)\sigma^{\frac{1}{2}}\,.$$

References

(1) A. Jenčová, Preservation of a quantum Rényi relative entropy implies existence of a recovery map, Journal of Physics A: Mathematical and Theoretical 50 (2017) 085303.
(2) A. Jenčová, Rényi relative entropies and noncommutative L_p-spaces, Annales Henri Poincare 19 (2018) 2513.
(3) F. Leditzky, C. Rouzé and N. Datta, Data processing for the sandwiched Rényi divergence: a condition for equality, Letters in Mathematical Physics 107 (2017) 61.
(4) D. Petz, Monotonicity of quantum relative entropy revisited, Reviews in Mathematical Physics 15 (2003) 79.
(5) M. M. Wilde, Optimised quantum f-divergences and data processing, Journal of Physics A: Mathematical and Theoretical 51 (2018) 374002.