Abstract

We extend the concept of (infinitesimal) Markovian divisibility from quantum channels to general linear maps and compact & convex sets of generators. We give a general approach for proving necessary criteria for (infinitesimal) Markovian divisibility. For infinitesimal Markovian divisible quantum channels, this yields, in any finite dimension \(d\), two upper bounds on the determinant: one with a \(\Theta(d)\)-power of the smallest singular value, and one with a product of \(\Theta(d)\) smallest singular values.

 Definitions

Markovian Divisibility - General

We start by introducing the two notions that are the focus of our work.

**Definition 1:** For \( G \subseteq \mathbb{C}^{d \times d} \) a set of matrices (called generators), we define
\[ D_G := \{ T \in M \mid \exists n \in \mathbb{N}, \text{ generators } (G_i)_{1 \leq i \leq n} \subseteq G \text{ s.t. } \sum_{i=1}^n G_i = T \}. \]
\( T_G \) is the set of linear maps that are Markovian divisible w.r.t. \( G \).

**Definition 2:** For \( G \subseteq \mathbb{C}^{d \times d} \) compact and convex with \( 0 \in M \), we define
\[ I_G := \{ T \in M \mid \forall \epsilon > 0 \exists n \in \mathbb{N}, \text{ generators } (G_i)_{1 \leq i \leq n} \subseteq G \text{ s.t. } (i \| G_i^\epsilon - I_G \| \leq \epsilon \text{ and } (i \| \sum_{i=1}^n G_i^\epsilon = T) \}. \]
\( T_G \) is the set of linear maps that are *infinitesimal* Markovian divisible w.r.t. \( G \).

Markovian Divisibility - Quantum

To apply our definitions to quantum channels, we need a suitable set of generators. We recall the following result due to [1].

**Theorem/Definition 3:** A linear map \( L : M \to M \) is the generator of a continuous dynamical semigroup of quantum channels if and only if it can be written in Lindblad form
\[ L(\rho) = i[\rho, H] + \sum_j L_j \rho L_j^\dagger. \]
where \( H = H^\dagger \in M \) and \( \{L_j\} \subseteq M \). Here, \( \{\cdot,\cdot\} \) denotes the anti-commutator.

When talking about (infinitesimal) Markovian divisible quantum channel, we mean this in the sense of Definitions 1 and 2, with \( G \) the set of Lindblad generators (in dimension \( d \)). [2] provides a characterization of infinitesimal (infinitesimal) Markovian divisible qubit channels. But no non-trivial necessary or sufficient criteria are known for higher dimensions.

Results

Determinant vs singular values - General

We want to understand under which assumptions on \( G \), any (infinitesimal) Markovian divisible map \( T \) has to satisfy an inequality of the form
\[ |\det(T)| \leq (\prod_{i=1}^n |s_i(T)|)^p. \]

We first show how to obtain such a singular value inequality for an (infinitesimal) Markovian divisible map from an eigenvalue inequality for generators.

**Theorem 4:** Let \( G \subseteq M \) be a set of generators. Let \( T \in T_G \) and suppose that every \( G \in \hat{G} \) satisfies \( |\det(G + G^\dagger) - p \sum_{i=1}^n s_i(G + G^\dagger)| \leq 0 \). Then \( |\det(T)| \leq (\prod_{i=1}^n |s_i(T)|)^p \).

**Corollary 5:** Let \( G \subseteq M \) be a compact and convex set of matrices containing \( 0 \in M \). Let \( G := \{ AG \mid \lambda \in [0,1], G \text{ an extreme point of } G \} \subseteq \hat{G} \). Assume that every \( G \in \hat{G} \) satisfies \( |\det(G + G^\dagger) - p \sum_{i=1}^n s_i(G + G^\dagger)| \leq 0 \). Let \( T \in T_G \). Then \( 0 \leq |\det(T)| \leq (\prod_{i=1}^n |s_i(T)|)^p \).

Proof Strategy

**Reduction to eigenvalues of generators**

We sketch the main steps in proving Theorem 4 and Corollary 5.

For Markovian divisibility (Theorem 4):
1. As both \( |\det(T)| \) and \( s_i(T) \) depend continuously on \( T \in M \), it suffices to prove the desired inequality for \( T \in D_G \) or for \( T \in T_G \).
2. By multiplicativity/submultiplicativity of the determinant/products of smallest singular values, it suffices to prove the inequality for single factors to get it for a product of maps.
3. Using that the singular values of a matrix exponential are majorized by the singular values of the matrix exponential of the real part, go from a singular value inequality for a matrix exponential to an eigenvalue inequality for the exponent.

For infinitesimal Markovian divisibility (Corollary 5):

Use an analogous line of reasoning. To reduce the assumption on the generators to an assumption on truncated rays through extreme points, use Krein-Milman and Trotterization.

Determinant vs smallest singular values - Quantum

We are particularly interested in obtaining criteria for (infinitesimal) Markovian divisible quantum channels. Based on Theorem 4 and Corollary 5, we can establish new necessary criteria in the form of upper bounds on the determinant via smallest singular values.

**Corollary 6:** Let \( T \in T_G \). Then \( 0 \leq |\det(T)| \leq (\prod_{i=1}^n |s_i(T)|)^p \). (Here, \( k = 1 \).

**Corollary 7:** Let \( T \in T_G \). Then \( 0 \leq |\det(T)| \leq (\prod_{i=1}^n |s_i(T)|)^p \). (Here, \( p = 1 \).

What are these criteria good for?

- Corollary 6 can be used to analytically construct, in any finite dimension, a set of channels that contains provably non infinitesimal Markovian divisible ones.
- For Corollary 7, we don’t know yet. Even for a conjectured strengthening of our result, we have not yet found explicit examples of channels that violate the inequality.

- Do you have suggestions for us?

Eigenvalues of Lindblad generators

By Theorem 4 and Corollary 5, we can get Corollaries 6 and 7 from eigenvalue inequalities for Lindblad generators. We can prove those using tools from matrix analysis [2].

The first such inequality, which yields Corollary 6, is

**Lemma 8:** Let \( L : M \to M \), \( \lambda L(\rho) = L_\lambda \rho L_\lambda^\dagger - \frac{1}{2}[L_\lambda^\dagger L_\lambda, \rho] \) be a purely dissipative Lindblad generator with one Lindbladian \( L \in M \). Then
\[ |\det(L_\lambda^\dagger L_\lambda - \frac{1}{2}[L_\lambda^\dagger L_\lambda, \rho])| \leq 0. \]

And to get Corollary 7, we use

**Lemma 9:** Let \( L : M \to M \), \( \lambda L(\rho) = L_\lambda \rho L_\lambda^\dagger - \frac{1}{2}[L_\lambda^\dagger L_\lambda, \rho] \) be a purely dissipative Lindblad generator with one Lindbladian \( L \in M \). Then for \( f(\rho) = 2d - 2\sqrt{d+1} \) we have
\[ |\det(L_\lambda^\dagger L_\lambda - \frac{1}{2}[L_\lambda^\dagger L_\lambda, \rho])| \leq 0. \]

References


[5] Technical University of Munich