Necessary Criteria for Markovian Divisibility of Linear Maps

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Matthias C. Caro and Benedikt Graswald

Technical University of Munich

Abstract

We extend the concept of (infinitesimal) Markovian divisibility from quantum channels [4] to general linear maps and compact & convex sets of generators. We give a general approach for proving necessary criteria for (infinitesimal) Markovian divisibility. For infinitesimal Markovian divisible quantum channels, this yields, in any finite dimension d, two upper bounds on the determinant: one with a $\Theta(d)$-power of the smallest singular value, and one with a product of $\Theta(d)$ smallest singular values.

Definitions

Markovian Divisibility - General

We start by introducing the two notions that are the focus of our work.

Definition 1: For $\mathcal{G} \subset \mathbb{C}^{d\times d}$ a set of matrices (called *generators*), we define

$$\mathcal{D}_{\mathcal{G}} := \{T \in \mathcal{M}_d \mid \exists n \in \mathbb{N},\ \text{generators } \{G_i\}_{1\leq i\leq n} \subset \mathcal{G} \text{ s.t. } \prod_{i=1}^{n} e^{G_i} = T\}.$$

$\overline{\mathcal{D}}_{\mathcal{G}}$ is the set of linear maps that are *Markovian divisible w.r.t.* $\mathcal{G}$.

Definition 2: For $\mathcal{G} \subset \mathbb{C}^{d\times d}$ compact and convex with $0 \in \mathcal{M}_d$, we define

$$\mathcal{I}_{\mathcal{G}} := \{T \in \mathcal{M}_d \mid \forall \varepsilon > 0\ \exists n \in \mathbb{N},\ \text{generators } \{G_j\}_{1\leq j\leq n} \subset \mathcal{G} \text{ s.t. } (i)\ \left\|e^{G_j} - \mathbb{1}_d\right\| \leq \varepsilon\ \forall j \text{ and } (ii) \prod_{j=1}^{n} e^{G_j} = T\}.$$

$\overline{\mathcal{I}}_{\mathcal{G}}$ is the set of linear maps that are *infinitesimal Markovian divisible w.r.t.* $\mathcal{G}$.

It is not hard to see that we can w.l.o.g. assume $\mathcal{G}$ to be compact without changing the class $\overline{\mathcal{I}}_{\mathcal{G}}$. Thus, we can also use Definition 2 for unbounded $\mathcal{G}$.

Markovian Divisibility - Quantum

To apply our definitions to quantum channels, we need a suitable set of generators. We recall the following result due to [2, 3]:

Theorem/Definition 3: A linear map $L : \mathcal{M}_d \to \mathcal{M}_d$ is the generator of a continuous dynamical semigroup of quantum channels if and only if it can be written in *Lindblad form*

$$L(\rho) = i[\rho, H] + \sum_j \mathcal{L}_j \rho \mathcal{L}_j^\dagger - \frac{1}{2}\{\mathcal{L}_j^\dagger \mathcal{L}_j, \rho\},$$

where $H = H^\dagger \in \mathcal{M}_d$ and $\{\mathcal{L}_j\}_j \subset \mathcal{M}_d$. Here, $\{\cdot,\cdot\}$ denotes the anti-commutator.

When talking about (infinitesimal) Markovian divisible quantum channel, we mean this in the sense of Definitions 1 and 2, with $\mathcal{G}$ the set of Lindblad generators (in dimension d). [4] provides a characterization of infinitesimal (infinitesimal) Markovian divisible qubit channels. But no non-trivial necessary or sufficient criteria are known for higher dimensions.

A "trivial" (i.e., easy-to-see) necessary criterion is non-negativity of the determinant.

Results

Determinant vs smallest singular values - General

We want to understand under which assumptions on $\mathcal{G}$, any (infinitesimal) Markovian divisible map T has to satisfy an inequality of the form

$$|\det(T)| \leq \left(\prod_{i=1}^{k} s_i^\uparrow(T)\right)^p.$$

We first show how to obtain such a singular value inequality for an (infinitesimal) Markovian divisible map from an eigenvalue inequality of generators.

Theorem 4: Let $\mathcal{G} \subseteq \mathcal{M}_d$ be a set of generators. Let $T \in \overline{\mathcal{D}}_{\mathcal{G}}$ and suppose that every $G \in \mathcal{G}$ satisfies $\mathrm{Tr}[G + G^*] - p\sum_{i=1}^{k} \lambda_i^\uparrow(G + G^*) \leq 0$. Then $|\det(T)| \leq \left(\prod_{i=1}^{k} s_i^\uparrow(T)\right)^p$.

Corollary 5: Let $\mathcal{G} \subset \mathcal{M}_d$ be a compact and convex set of matrices containing $0 \in \mathcal{M}_d$. Let $\tilde{\mathcal{G}} := \{\lambda G \mid \lambda \in [0,1],\ G \text{ an extreme point of } \mathcal{G}\} \subset \mathcal{G}$. Assume that every $\tilde{G} \in \tilde{\mathcal{G}}$ satisfies $\mathrm{Tr}[\tilde{G} + \tilde{G}^*] - p\sum_{i=1}^{k} \lambda_i^\uparrow(\tilde{G} + \tilde{G}^*) \leq 0$. Let $T \in \overline{\mathcal{I}}_{\mathcal{G}}$. Then $0 \leq \det(T) \leq \left(\prod_{i=1}^{k} s_i^\uparrow(T)\right)^p$.

Determinant vs smallest singular values - Quantum

We are particularly interested in obtaining criteria for (infinitesimal) Markovian divisible quantum channels. Based on Theorem 4 and Corollary 5, we can establish new necessary criteria in the form of upper bounds on the determinant via smallest singular values.

Corollary 6: Let $T \in \overline{\mathcal{I}}_d$. Then $0 \leq \det(T) \leq \left(s_1^\uparrow(T)\right)^{\frac{d}{2}}$. (Here, "$k = 1$".)

Corollary 7: Let $T \in \overline{\mathcal{I}}_d$. Then $0 \leq \det(T) \leq \prod_{i=1}^{\lfloor 2d - 2\sqrt{2d} + 1\rfloor} s_i^\uparrow(T)$. (Here, "$p = 1$".)

What are these criteria good for?

- Corollary 6 can be used to analytically construct, in any finite dimension, a set of channels that contains provably non infinitesimal Markovian divisible ones.
- For Corollary 7, we don't know yet. Even for a conjectured strengthening of our result, we have not yet found explicit examples of channels that violate the inequality.
  $\to$ Do you have suggestions for us?

Proof Strategy

Reduction to eigenvalues of generators

We sketch the main steps in proving Theorem 4 and Corollary 5.

For Markovian divisibility (Theorem 4):

1. As both $|\det(T)|$ and $s_i^\uparrow(T)$ depend continuously on $T \in \mathcal{M}_d$, it suffices to prove the desired inequality for $T \in \mathcal{D}_{\mathcal{G}}$ or for $T \in \mathcal{I}_{\mathcal{G}}$.
2. By multiplicativity/submultiplicativity of the determinant/products of smallest singular values, it suffices to prove the inequality for single factors to get it for a product of maps.
3. Using that the singular values of a matrix exponential are majorized by the singular values of the matrix exponential of the real part, go from a singular value inequality for a matrix exponential to an eigenvalue inequality for the exponent.

For infinitesimal Markovian divisibility (Corollary 5):

Use an analogous line of reasoning. To reduce the assumption on the generators to an assumption on truncated rays through extreme points, use Krein-Milman and Trotterization.

Eigenvalues of Lindblad generators

By Theorem 4 and Corollary 5, we can get Corollaries 6 and 7 from eigenvalue inequalities for Lindblad generators. We can prove those using tools from matrix analysis [1].

The first such inequality, which yields Corollary 6, is

Lemma 8: Let $L : \mathcal{M}_d \to \mathcal{M}_d$, $L(\rho) = \mathcal{L}\rho\mathcal{L}^\dagger - \frac{1}{2}\{\mathcal{L}^\dagger\mathcal{L}, \rho\}$ be a purely dissipative Lindblad generator with one Lindbladian $\mathcal{L} \in \mathcal{M}_d$. Then

$$\mathrm{Tr}[L + L^*] - \frac{d}{2}\Lambda_1^\uparrow(L + L^*) \leq 0.$$

And to get Corollary 7, we use

Lemma 9: Let $L : \mathcal{M}_d \to \mathcal{M}_d$, $L(\rho) = \mathcal{L}\rho\mathcal{L}^\dagger - \frac{1}{2}\{\mathcal{L}^\dagger\mathcal{L}, \rho\}$ be a purely dissipative Lindblad generator with one Lindbladian $\mathcal{L} \in \mathcal{M}_d$. Then for $f(d) = 2d - 2\sqrt{2d} + 1$ we have

$$\mathrm{Tr}[L + L^*] - \sum_{K=1}^{\lfloor f(d)\rfloor} \Lambda_K^\uparrow(L + L^*) \leq 0.$$

Lemmas 8 and 9 are almost optimal. Both times, the dependence on d cannot be better than linear (in leading order), and we even get close to the optimal prefactors.

References

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