Krylov subspace type methods for the computation of non-negative or sparse solutions of ill-posed problems.

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An important problem

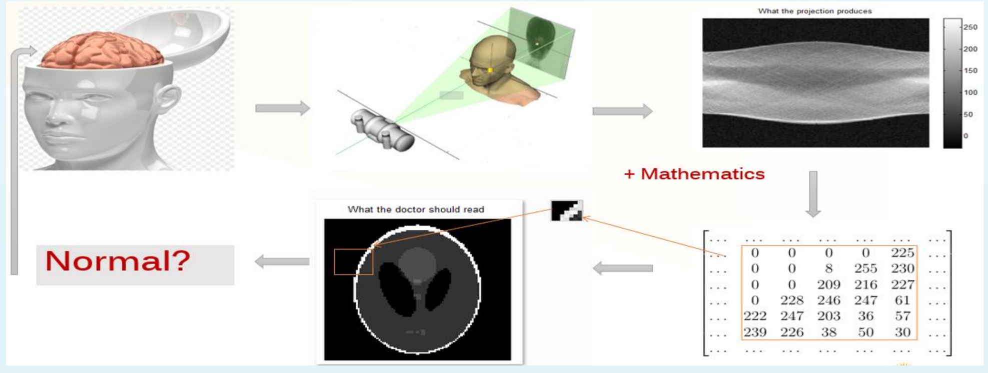
Many applications in science and engineering require the solution of least-squares problems of the form

$$\min_{x\in R^n} \| Ax-b \|_2 \quad (1)$$

- ❑ $A\in R^{m\times n}$ is a large matrix, whose singular values "cluster" at the origin.
- ❑ The vector $b\in R^m$ represents the measured data that is contaminated by errors, $b=b\_true+e$
- ❑ $e\in R^m$ represents the error vector that may stem from measurements or other factors.
- ❑ b_true represents the error free unknown vector.
- ❑ The observations are collected from indirect measurements, often incomplete and use the data to obtain a target of interest.
- ❑ Problems with the above properties that have many solutions, no solution or stability problems are called Ill-posed inverse problems.

Motivation and Goals

- ❑ In the signal processing field scientists would like to represent the signal with as few elementary components as possible, without significantly affecting the quality of the reconstruction.
- ❑ Biomedical Scientists would like to be able to minimize the amount of radiation going through the body during MRI and CT. Reducing the number of radiation yields loss of information.
- ❑ Neuroscientists would like to study group testing in sensory systems, sparse (multidimensional) neural coding and sparse network interactions.
- ❑ NASA would like to deblur images coming from space.



Our main goals are:

- ➢ Impose nonnegativity to the computed approximate solution in order to improve the reconstruction when the desired solution is known to be nonnegative. ✓
- ➢ Impose Sparsity to find the solution with the smallest norm in order to minimize the number of nonzero components in the desired solution. ✓
- ➢ Prove the convergence of the proposed method. ✓

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Methodology

Sparse reconstruction plays an important role in many application such as signal and image processing, compressed sensing, model selection and many others.
To obtain the sparsest solution, one naturally impose to solve the following minimization problems:

$$\min_{x\in R^n}\{\mu \|x\|_1 + \frac{1}{2\delta}\|x\|_2^2 : Ax=b\},\ \mu,\delta\text{-user defined constants} \quad (2)$$

The goal of the linearized Bregman iteration is to find an approximate solution of (1) of minimal '$l1$' norm, i.e. to approximately solve (2).

$$\begin{cases} v^{(k+1)} = v^{(k)} + A^T(Au^{(k)}-b) \\ s^{(k+1)} = \delta T_\mu(V_k z^{(k+1)}) \end{cases} \quad (3)$$
$$\boldsymbol{v^{(0)} = s^{(0)} = 0}$$

In order to improve the quality of the reconstruction and the speed of the convergence, we propose:

1. Projected Linearized Bregman approach (PLB)

- ▪ Projection to a Krylov subspace with a relatively small dimension using k Golub-Kahan bidiagonalizations that generates an orthonormal basis for the Krylov subspace of dimension k given by:
$$K_k(A^TA, A^Tb) = span\{A^Tb, A^TAA^Tb, \dots, (A^TA)^{k-1}A^Tb\}.$$
- ▪ The Linearized Bregman Iterations for (3) become:

$$\begin{cases} z^{(k+1)} = z^{(k)} + B_{k+1}^T(\||b|\|e_1 - B_{k+1}V_k^Ts^{(k)}) \\ s^{(k+1)} = \delta T_\mu(V_k z^{(k+1)}) \end{cases} \quad (4)$$

- ▪ Projection to the nonnegative cone by making zero all the nonnegative elements. Define the set Ω_0={x,x≥ 0} and $P_{\Omega_0} = [p_{\Omega_0}(z(1)) \dots, p_{\Omega_0}(z(n))]$.

Once the projection is applied, then the iterations become:

$$\begin{cases} z^{(k+1)} = z^{(k)} + B_{k+1}^T(\||b|\|e_1 - B_{k+1}V_k^Ts^{(k)}) \\ s^{(k+1)} = \delta P_{\Omega_W}(T_\mu(V_k z^{(k+1)})) \\ z^{(0)} = s^{(0)} = 0. \end{cases} \quad (5)$$

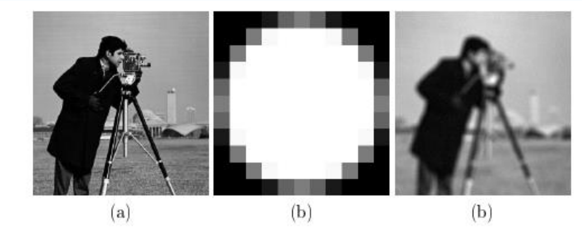
2. Further Improvement: Accelerated PLB approach

The proposed method PLB is computationally cheap and improves the quality of the desired solution, but it might require a large number of iterations.
We propose another schema to speed up the convergence by introducing the following iterations.

$$\begin{cases} v^{(k+1)} = z^{(k)} - A^T(Au^{(k)}-b) \\ z^{(k+1)} = \alpha_k v^{(k+1)} + (1-\alpha_k)v^{(k)} \\ u^{(k+1)} = \delta P_{\Omega_W}(T_\mu(z^{(k+1)})) \end{cases} \quad (6)$$

Results and Examples

Example 1: Cameraman test problem with out of focus blur and noise contaminated.



Cameraman test problem: (a) Uncontaminated image (246 × 246 pixels), (b) the PSF (11 × 11 pixels), (c) blur and noise contaminated image ($\epsilon = 0.02\|b\|$)

	Noise level						
Method	0.001	0.002	0.005	0.01	0.05	0.1	0.15
Bregman	**0.1014**	**0.1014**	**0.1015**	**0.1098**	**0.1522**	**0.1587**	**0.1608**
Fista	0.1091	0.1129	0.1015	0.1155	0.1511	0.1654	0.1673
IR	0.1512	0.1511	0.1516	0.1528	0.1753	0.2085	0.3651
IRN	0.1505	0.1506	0.1509	0.1527	0.1648	0.182	0.2031
IRNhtv	0.1481	0.1482	0.1486	0.149	0.1591	0.1665	0.1724

Table: Cameraman test problem: RRE for each noise level and for each method.

	Time (s) for each noise level						
Method	0.001	0.002	0.005	0.01	0.05	0.1	0.15
Bregman	**0.1121**	**0.1291**	**0.1206**	**0.1364**	**0.121**	**0.1237**	**0.1318**
Fista	2.4861	2.6563	1.375	0.9051	0.6268	0.4853	0.3426
IR	3.7271	3.7433	3.7715	3.7408	3.7069	0.2866	0.1825
IRN	10.5155	10.3165	11.4531	10.5311	10.0519	10.1045	10.2336
IRNhtv	34.6592	34.6987	35.6659	34.9437	33.9198	34.8044	35.5324

Table: Cameraman test problem: Time(s) for each noise level.

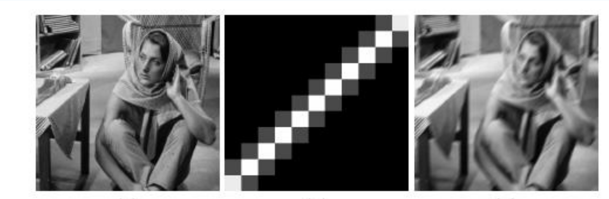
Figure: Cameraman test image restoration: (a) PLB, (b)Fista , (c) Irn

We compare our proposed method with Fista, IR, IRN and IRNHtv. Our method reconstructs with higher quality and faster.

Example 2: Barbara test problem with motion blur and noise contaminated

	Noise level						
	0.001	0.002	0.005	0.01	0.05	0.1	0.15
Bregman	**0.0832**	**0.0849**	**0.0887**	**0.0912**	**0.1126**	**0.1221**	**0.1302**
Fista	0.1623	0.1712	0.1917	0.0999	0.1144	0.1231	0.1334
IR	0.1376	0.1366	0.1369	0.1379	0.1607	0.1932	0.2182
IRN	0.1369	0.1359	0.1362	0.1372	0.1602	0.1932	0.2177
IRNhtv	0.1336	0.1346	0.1348	0.1353	0.1447	0.1531	0.1592

Barbara test problem: RRE for each noise level and for each tested method.



Barbara test problem: (a) Uncontaminated image (246 × 246 pixels), (b) the PSF (11 × 11 pixels), (c) blur and noise contaminated image ($\epsilon = 0.01\|b\|$)

	Time (s) for each noise level						
	0.001	0.002	0.005	0.01	0.05	0.1	0.15
Bregman	**0.318**	**0.351**	**0.309**	**0.299**	**0.367**	**0.369**	**0.373**
Fista	8.123	7.391	8.263	2.023	0.794	0.708	0.702
IR	10.23	11.9	12.085	12.254	12.061	12.18	12.101
IRN	12.35	12.136	11.061	11.110	11.193	11.44	11.09
IRNhtv	176.53	175.765	177.268	178.817	177.281	177.63	178.27

Barbara test problem: Time in seconds needed for each method for each noise level.

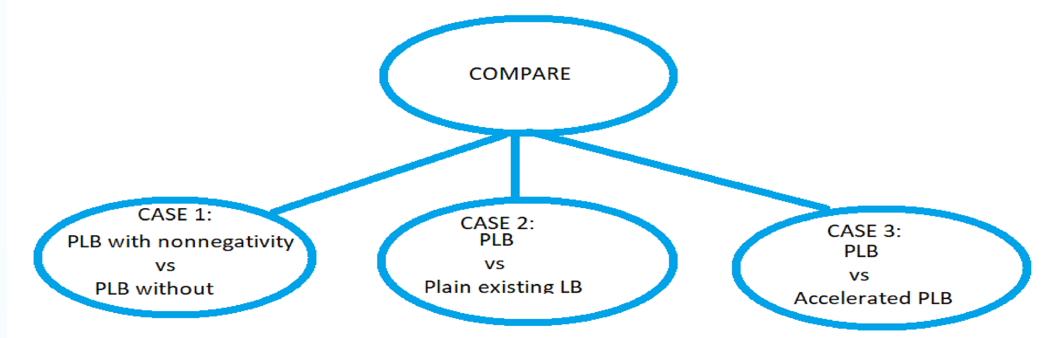
Barbara test image restoration: (a) PLB, (b)Fista , (c) Irn

We claim that the PLB algorithm is computationally cheaper and faster than the Linearized Bregman method. Our numerical tests that support our claim.

Take home conclusions

- ❑ We introduce a new method for solving discrete inverse problems.
- ❑ First we projected our data to a smaller subspace and then we apply the linear Bregman iterations to find an approximate solution of the true solution.
- ❑ We observed that by reducing the solution subspace the linearized Bregman iteration are computationally cheaper.
- ❑ Imposing the nonnegativity constraints for some applications where the solution is known to be nonnegative improved the RRE compared to the non-constraint case.
- ❑ Based on all the tests that we did we observed that when the noise norm is increased, the RRE of Projected linearized Bregman solution does not grow fast.
- ❑ This is an advantage of PLB method and it gives restoration of higher quality when the noise level is high compared to other considered method.
- ❑ Our approach is computationally cheaper and faster then the compared methods.
- ❑ The quality of the reconstructed solution is improved.

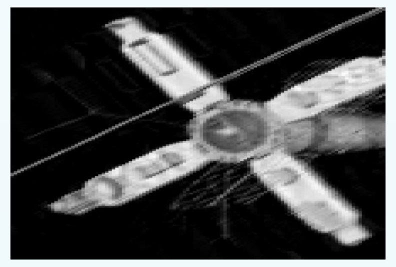
Comparison



CASE 1: Our first comparison aims to show that projecting to the nonnegative cone has practical improvements in the quality of the restored solution.

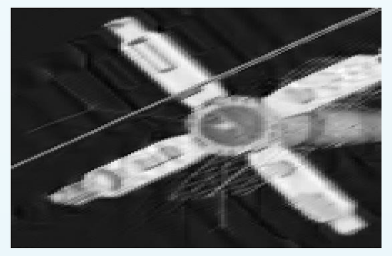
	PLB on Satellite				
	RRE	PLB it.	Total time	Krylov it.	Krylov time
1%	0.1276	300	58.75	16	1.4
5%	0.2033	98	19.32	6	0.468
15%	0.2618	29	5.921	3	0.125

Results from PLB method for different noise levels. We compare RRE, iterations that PLB need to complete, the dimesion of Krylov subspace and the time needed to generate the Krylov subspace.



	PLBn on Satellite				
	RRE	PLBn it.	Total time	Krylov it.	Krylov time
1%	0.1337	300	60.56	16	1.25
5%	0.2126	87	17.56	6	0.359
15%	0.2703	23	4.656	3	0.187

Results from PLBn method for different noise levels. We compare RRE, iterations that PLB need to complete, the dimesion of Krylov subspace and the time needed to generate the Krylov subspace.



CASE 2: We show that our approach PLB method is faster and cheaper than plain existing LB method.

	PLB		
	RRE	Total iterations	Time (seconds)
Satellite 5%	0.2033	98	19.32
Satellite 15%	0.2618	29	5.921

RRE, Total number of iterations and the time in seconds needed by PLB method for different noise levels.

	LB		
	RRE	Total iterations	Time (seconds)
Satellite 5%	0.2908	300	25.543
Satellite 15%	0.323	300	34.681

RRE, Total number of iterations and the time in seconds needed by LB method for different noise levels.

CASE 3: We show numerical examples supporting the fact that the acceleration approach speeds up the convergence and improves the quality of the restoration.

	RRE	Iterations	Total time	Krylov It	Krylov time
PLB	0.2212	346	58.09	23	1.8438
Accelerated PLB	**0.2169**	**132**	**26.5**	**23**	**0.4646**

Comparing PLB and Accelerated PLB

References

Some selected references:

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