Boundary and VSL Control for Large-Scale Urban Traffic Networks

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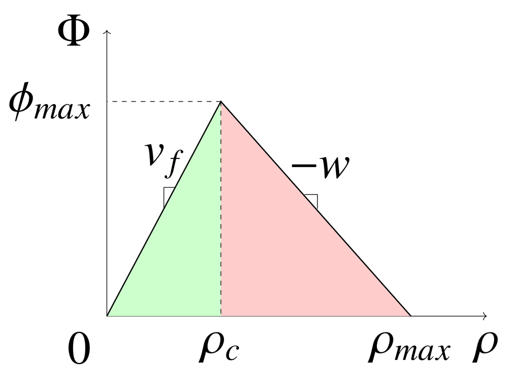
Motivation

Our goal is to elaborate a general **explicit** method to design controllers for large-scale networks.

Traffic's evolution in some *large urban network* can be described by a 2D conservation law model such as **2D LWR** (macroscopic approach). It describes traffic density $\rho(x, y, t)$ on a 2D-plane Ω as:

$$\frac{\partial \rho(x, y, t)}{\partial t} + \nabla \cdot \vec{\Phi}(x, y, \rho) = 0, \qquad (1)$$

- $\vec{\Phi} = \Phi(x, y, \rho)\vec{d}_\theta(x, y)$: flow vector function
- flow direction: $\vec{d}_\theta(x, y) = (\cos(\theta(x, y)) \ \sin(\theta(x, y)))^T$
- magnitude Φ is a concave function of ρ (FD)

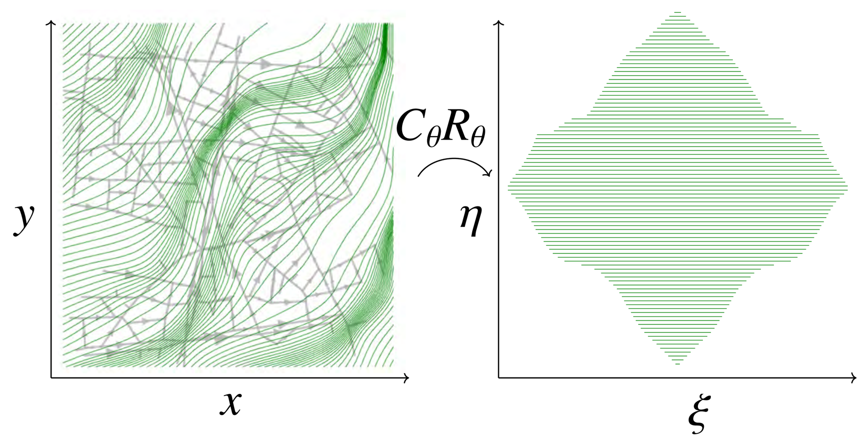


We define $\vec{d}_\theta$, ρ_{max}, v_f and w $\forall (x, y) \in \Omega$ by interpolation methods. *Network can not contain loops!*

Our contributions

- A method of transforming a 2D-LWR into a continuous set of 1D systems to simplify control.
- First explicitly derived controllers acting in large-scale networks: 1) *boundary* controller tracking a *space- and time-dependent trajectory with shocks*, and 2) *VSL* controller achieving any *space-dependent* desired equilibrium.

Coordinate Transformation



Define a **coordinate transformation** translating integral curves of $\vec{d}_\theta$ into a set of straight parallel lines:

$$\begin{pmatrix} d\xi \\ d\eta \end{pmatrix} = C_\theta(x, y) R_\theta(x, y) \begin{pmatrix} dx \\ dy \end{pmatrix},$$

where R_θ and C_θ are rotation and scaling matrices:

$$R_\theta(x, y) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad C_\theta(x, y) = \begin{pmatrix} \alpha(x, y) & 0 \\ 0 & \beta(x, y) \end{pmatrix}$$

where α and β are scaling parameters that satisfy:

$$\sin\theta \frac{\partial(\ln\alpha)}{\partial x} + \cos\theta \frac{\partial(\ln\alpha)}{\partial y} = \cos\theta \frac{\partial\theta}{\partial x} + \sin\theta \frac{\partial\theta}{\partial y},$$
$$\cos\theta \frac{\partial(\ln\beta)}{\partial x} + \sin\theta \frac{\partial(\ln\beta)}{\partial y} = \sin\theta \frac{\partial\theta}{\partial x} - \cos\theta \frac{\partial\theta}{\partial y}.$$

Introduce $\bar{\rho} = \rho/\alpha\beta$, $\bar{\Phi} = \Phi/\beta$ and rewrite (1) as:

A continuous set of 1D-LWR equations

$$\frac{\partial \bar{\rho}(\xi, \eta, t)}{\partial t} + \frac{\partial(\bar{\Phi}(\xi, \eta, \bar{\rho})}{\partial \xi} = 0,$$

where η parametrizes the flow path.

Boundary Control: Problem

Define IBVP in (ξ, η)-space:

$$\begin{cases} \frac{\partial \bar{\rho}(\xi, \eta, t)}{\partial t} + \frac{\partial(\bar{\Phi}(\xi, \eta, \bar{\rho})}{\partial \xi} = 0, \\ \bar{\phi}_{in}(\eta, t) = \min\left(\bar{u}_{in}(\eta, t), \bar{S}\left(\bar{\rho}\left(\xi_{min}(\eta), \eta, t\right)\right)\right), \\ \bar{\phi}_{out}(\eta, t) = \min\left(\bar{D}\left(\bar{\rho}\left(\xi_{max}(\eta), \eta, t\right)\right), \bar{u}_{out}(\eta, t)\right), \\ \bar{\rho}(\xi, \eta, 0) = \bar{\rho}_0(\xi, \eta), \end{cases} \qquad (2)$$

where $\bar{D}(\bar{\rho})$, $\bar{S}(\bar{\rho})$ are demand and supply functions:

$$D(\rho) = \begin{cases} \phi(\rho), & \rho \in [0, \rho_c], \\ \phi_{max}, & \rho \in (\rho_c, \rho_{max}], \end{cases} \qquad S(\rho) = \begin{cases} \phi_{max}, & \rho \in [0, \rho_c], \\ \phi(\rho), & \rho \in (\rho_c, \rho_{max}]. \end{cases}$$

Problem 1

Design $\forall(\eta, t) \in \Omega \times \mathbb{R}^+$ boundary control laws $u_{in}(\eta, t)$ and $u_{out}(\eta, t)$ such that the vehicle density from (2) tracks a desired trajectory as $t \to \infty$.

Remark 1

Notice that controls $\bar{u}_{in}$ and $\bar{u}_{out}$ are not always accepted by the system. We use the viability solution in Hamilton-Jacobi formulation to analyse that.

Boundary Control: Result

Assumption 1. In- and outflows are lower than the minimal capacity during some time interval and

$$\bar{\phi}_{in}(\eta, t) \leq \min \bar{\phi}_{max}(\eta), \quad \bar{\phi}_{out}(\eta, t) \leq \min \bar{\phi}_{max}(\eta).$$

Assumption 2. Initial conditions have left (2).

Theorem 1

Under Assumptions 1 and 2, for any $\eta \in \Omega$, given the desired density $\bar{\rho}_d(\xi, t)$ and boundary flows $\bar{\phi}_{in_d}(t)$, $\bar{\phi}_{out_d}(t)$ as in (2), the controls

$$(1)\ \bar{u}_{in}(t) = \bar{\phi}_{in_d}(t) - ke(t), \qquad t \in \mathbb{R}^+$$
$$(2)\ \bar{u}_{out}(t) = \bar{\phi}_{out_d}(t) + ke(t),$$

$$\text{where } e(t) = \int_{\xi_{min}}^{\xi_{max}} \left(\bar{\rho}(\hat{\xi}, t) - \bar{\rho}_d(\hat{\xi}, t)\right) d\hat{\xi} \ \text{ and } \ k > 0,$$

provide that

$$\forall a, b \in [\xi_{min}, \xi_{max}] \quad \lim_{t \to \infty} \int_a^b \left(\rho(\hat{\xi}, t) - \rho_d(\hat{\xi}, t)\right) d\hat{\xi} = 0.$$

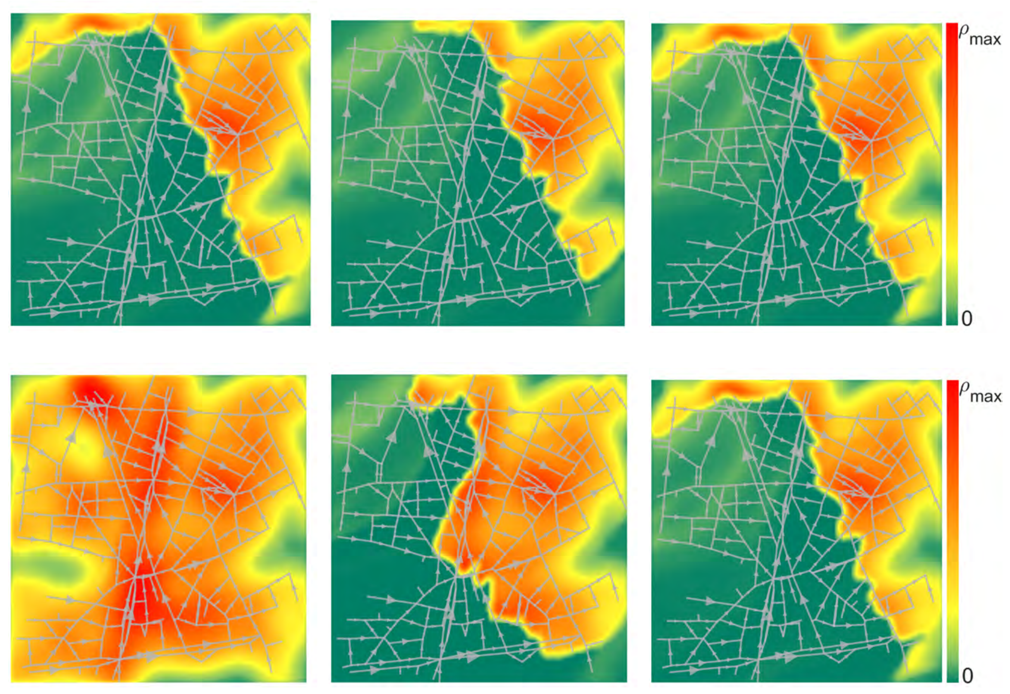
Remark 2

Integral convergence in Theorem 1 implies $\rho \approx \rho_d$ as $t \to \infty$, since a and b can be arbitrarily close.

Boundary Controller: Example

Grenoble downtown: desired time-periodic state (up) and controlled state with $k = 5 \cdot 10^{-5}$ (down).

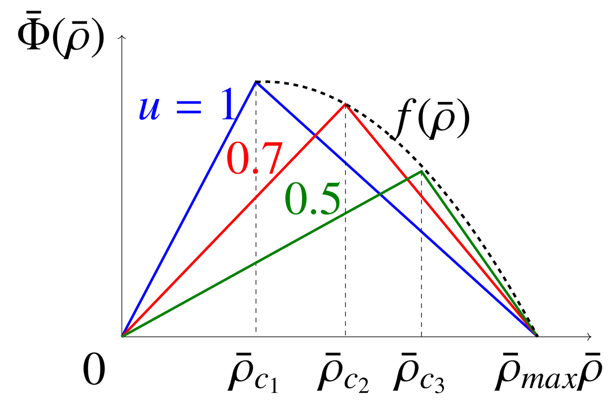
$t = 0$ $\quad t = 0.5\tau$ $\quad t = 2\tau$



VSL Control: Problem

Consider VSL-dependent FD $\bar{\Phi}(\xi, \eta, \bar{\rho}, u)$, where $u(\xi, \eta, t) \in [0, 1]$ is the ratio of imposed speed limit to the free-flow speed: no speed limit if $u = 1$, no movement if $u = 0$.

Real data showed that speed limits can increase critical density and enhance flow in congested regime.



$f(\xi, \eta, \bar{\rho})$ is the maximum flow over all VSL values:

$$f(\xi, \eta, \bar{\rho}) = \max_{u \in [0,1]} \bar{\Phi}(\xi, \eta, \bar{\rho}, u).$$

Problem 2

$\forall(\xi, \eta, t) \in \Omega \times \mathbb{R}^+$ find $u(\xi, \eta, t)$ such that:

$$\lim_{t \to \infty} (\tilde{\bar{\rho}}(\xi, \eta, t)) = 0,$$

where $\tilde{\bar{\rho}}$ is deviation from $\bar{\rho}^*(\xi, \eta) \in (0, \bar{\rho}_{max}(\xi, \eta))$.

VSL Control: Result

Introduce a set $G(\xi, \eta, \bar{\rho}, \bar{\phi})$, which is the inverse function of FD wrt u:

$$G(\xi, \eta, \bar{\rho}, \bar{\phi}) = \{u \in (0, 1] : \bar{\Phi}(\xi, \eta, \bar{\rho}, u) = \bar{\phi}\}.$$

Theorem 2

Let $u(\xi, \eta, t)$ be given $\forall(\xi, \eta, t) \in \Omega \times \mathbb{R}^+$ by:

$$u \in G(\bar{\rho}, \bar{\phi}), \quad \text{with} \quad \bar{\phi} = B \min_{\xi} \frac{f(\bar{\rho})}{B} \quad \text{and}$$

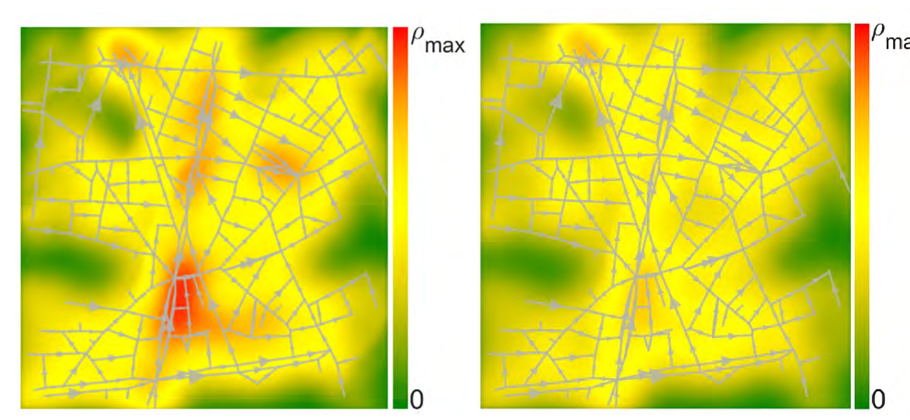
$$B(\xi, \eta, t) = 1 + \gamma \int_{\xi_{min}(\eta)}^{\xi} \tilde{\bar{\rho}}(\hat{\xi}, \eta, t) d\hat{\xi}, \ \text{with } \gamma \text{ s.t. B} > 0.$$

Then for every $\bar{\rho}_0(\xi, \eta) \in C^1(\Omega)$ the system has a unique solution $\bar{\rho}(\xi, \eta, t) \in C^1(\Omega)$ that asymptotically converges to the desired state.

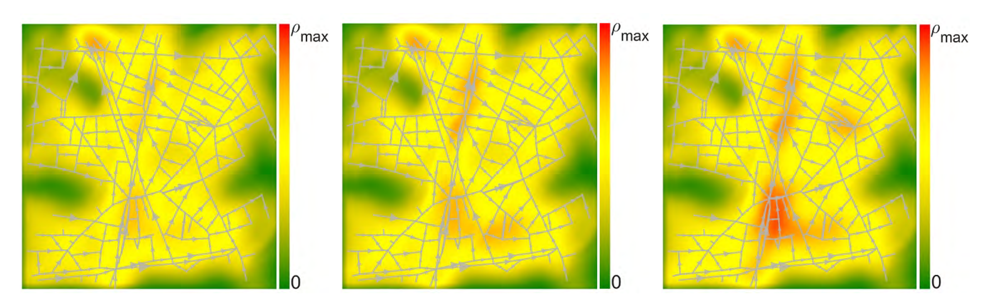
VSL Control: Example

As ρ^* we chose an optimal equilibrium corresponding to the throughput maximization for the maximal possible density, i.e. more cars will pass Grenoble at maximum flow that depends only on topology.

Optimal equilibrium $\quad$ Initial state



$t = 10\ min$ $\quad t = 30\ min$ $\quad t = 2\ hours$



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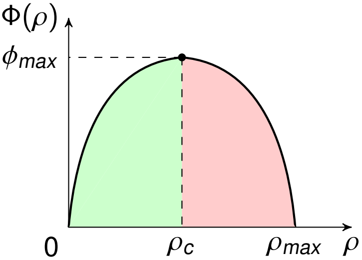
Macroscopic traffic modelling

2D-LWR model

$$\begin{cases} \dfrac{\partial \rho(x,y,t)}{\partial t} + \nabla \cdot \vec{\Phi}(x,y,\rho) = 0, \\ \rho(x,y,0) = \rho_0(x,y). \end{cases}$$

- $\rho(x,y,t)$: vehicle density
- $\vec{\Phi} = \Phi(x,y,\rho)\vec{d}_\theta(x,y)$: flow function

flow direction $\vec{d}_\theta(x,y)$

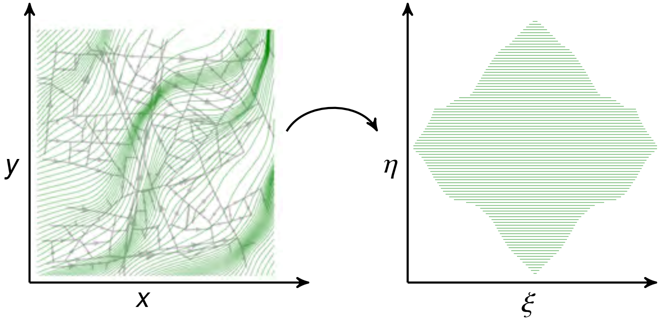


Objectives

Problem Formulation

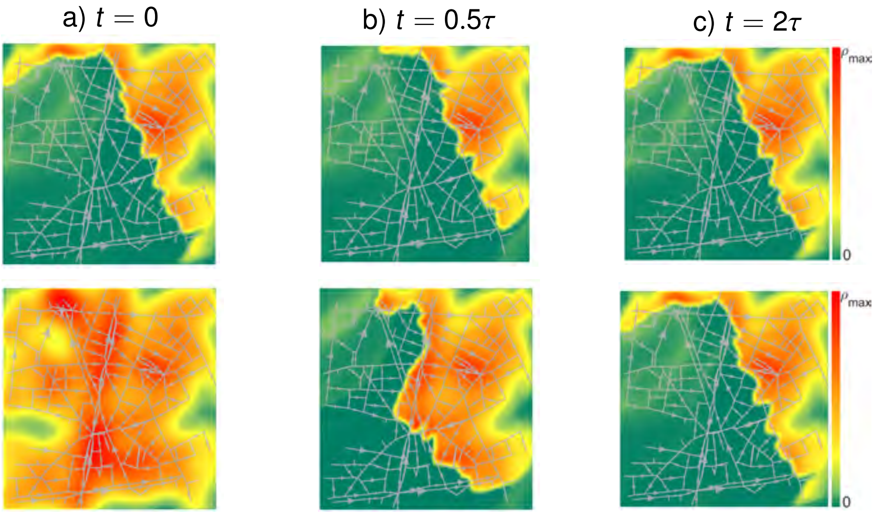
Given network topology with infrastructure parameters (maximal speeds and capacity for all roads) and a density from 2D-LWR model, design a **scale-free model-based** control technique to achieve any desired state.

Main method: transform integral curves of the flow field as below.



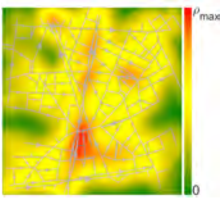
Boundary control

Grenoble downtown: desired state (up) and a feedback-controlled state (down)

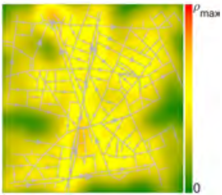


VSL control

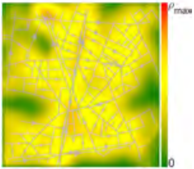
Equilibrium: throughput maximization for the maximal possible density



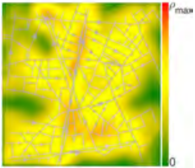
Desired state



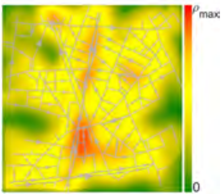
Initial state



$t = 10$ *min*



$t = 30$ *min*



$t = 2$ *hours*