Second Order Traffic Flow Networks: A New Modeling Approach

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Introduction

We consider the **Aw-Rascle-Zhang model** for traffic flow on uni-directional road networks.
It combines a conservation law for the density ρ with an additional conservation law for the mean traffic speed v. On each road i of the network, we require the following system to hold:

$$\partial_t \begin{pmatrix} \rho_i \\ \rho_i w_i \end{pmatrix} + \partial_x \begin{pmatrix} \rho_i v_i \\ \rho_i w_i v_i \end{pmatrix} = 0, \quad w_i = v_i + p_i(\rho_i) \tag{1}$$

where ρ_i and v_i are the density and the velocity on road i, respectively. With $p_i(\rho)$, we denote the pressure function and w_i is a Lagrangian marker.

We particularly focus on a novel approximation to the **homogenized pressure** by introducing an additional equation for the propagation of a reference pressure starting from the **junctions of the network**. The resulting system of coupled conservation laws is then solved using an appropriate numerical scheme of Godunov type. Numerical simulations compare our approach to other coupling conditions.

Homogenization at the junctions of networks

Any change in the value w_j at a junction induces also a change in the corresponding pressure p_j. This leads to coupling conditions at junctions that not only prescribe coupling or boundary conditions in terms of $(\rho, \rho v)$ but also in the form of the pressure p [1].
This interpretation is also consistent with a discretization in Lagrangian coordinates leading to the so called "follow-the-leader model". Cars passing trough a junction k and entering an outgoing road j have the average property $\overline{w_j}$ associated with the Young measure μ_x describing the mixture of cars.

$$\overline{w_j} := \int w \mathrm{d}\mu_x(w) = \vec{\beta}\vec{w}^T$$

where $\vec{\beta} = (\beta_{ij})_{i\in\delta_k^-}$ is a priority vector and $\vec{w} = (w_{i,0})_{i\in\delta_k^-}$ are the Lagrangian markers at $t = 0$. Assume that the pressure on an outgoing road in the traffic network is initially given by $p_{j,0}(\rho)$. The unique weak entropy solution of (1) is characterized by the relation

$$\tau = \sum_{i\in\delta_k^-} \beta_{ij} P_{j,0}^{-1}(w_{i,0} - v) =: (P_{j,0}^*)^{-1}(\overline{w_j} - v) \qquad \text{with } \tau = \frac{1}{\rho},\; P_{j,0}(\tau) = p_{j,0}\left(\frac{1}{\rho}\right)$$

The pressure $P_{j,0}^*(\tau)$ is, in fact, redefined such that for each τ, the velocity v is the velocity of the weak entropy solution of the Lagrangian formulation of (1). The homogenized pressure and the Lagrangian marker are defined as follows:

$$p_j^*(\rho) = P_j^*(1/\rho), \qquad \overline{w_j} = v + p_j^*(\rho). \tag{2}$$

For different values $w \neq \overline{w_j}$, a different pressure p^* is obtained. We approximate the homogenized pressure (2):

$$p^*(\rho) \approx p^{**}(\rho) = c(\vec{\beta}, \vec{w}) p_{j,0}(\rho). \tag{3}$$

System with adapted pressure

Assuming the pressure law is of the type $p(\rho) = c(\vec{\beta}, \vec{w})p_{j,0}(\rho)$ with a value $c(\vec{\beta}, \vec{w})$ independent of ρ, the pressure propagates with the velocity v of the cars, according to equation (1). However, the value of c might change over time due to the coupling at the junction. This will change the value of c corresponding to the value of the homogenized Lagrangian marker $\overline{w_j}$. Once this marker enters the outgoing road it is transported with the flow.

We propose the **adapted pressure (AP) model**

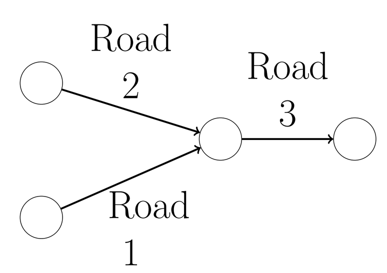
$$\partial_t \begin{pmatrix} \rho \\ \rho w \\ \rho c \end{pmatrix} + \partial_x \begin{pmatrix} \rho v \\ \rho w v \\ \rho c v \end{pmatrix} = 0.$$

with eigenvalues $\lambda_1 = v - p'(\rho)\rho < v = \lambda_2 = \lambda_3$.

Lemma 1. *Consider a junction k with n incoming roads, a single outgoing road, with constant initial data $U_i = (\rho_{i,0}, w_{i,0})$, $i \in (\delta_k^- \cup \delta_k^+)$, initial pressure $p(\rho) = c_{j,0}\,\rho^\gamma, \gamma \geq 1$ on the outgoing road $j = n + 1$. There exists a unique network solution which conserves the mass and momentum at the junction using our approximation of the homogenized pressure on the outgoing road*

$$p_j^{**}(\rho) = c_{j,0}\left(\sum_{i=1}^n \beta_i w_i \left(\sum_{l=1}^n \frac{\beta_l}{w_l^{1/\gamma}}\right)\right)^\gamma \rho^\gamma.$$

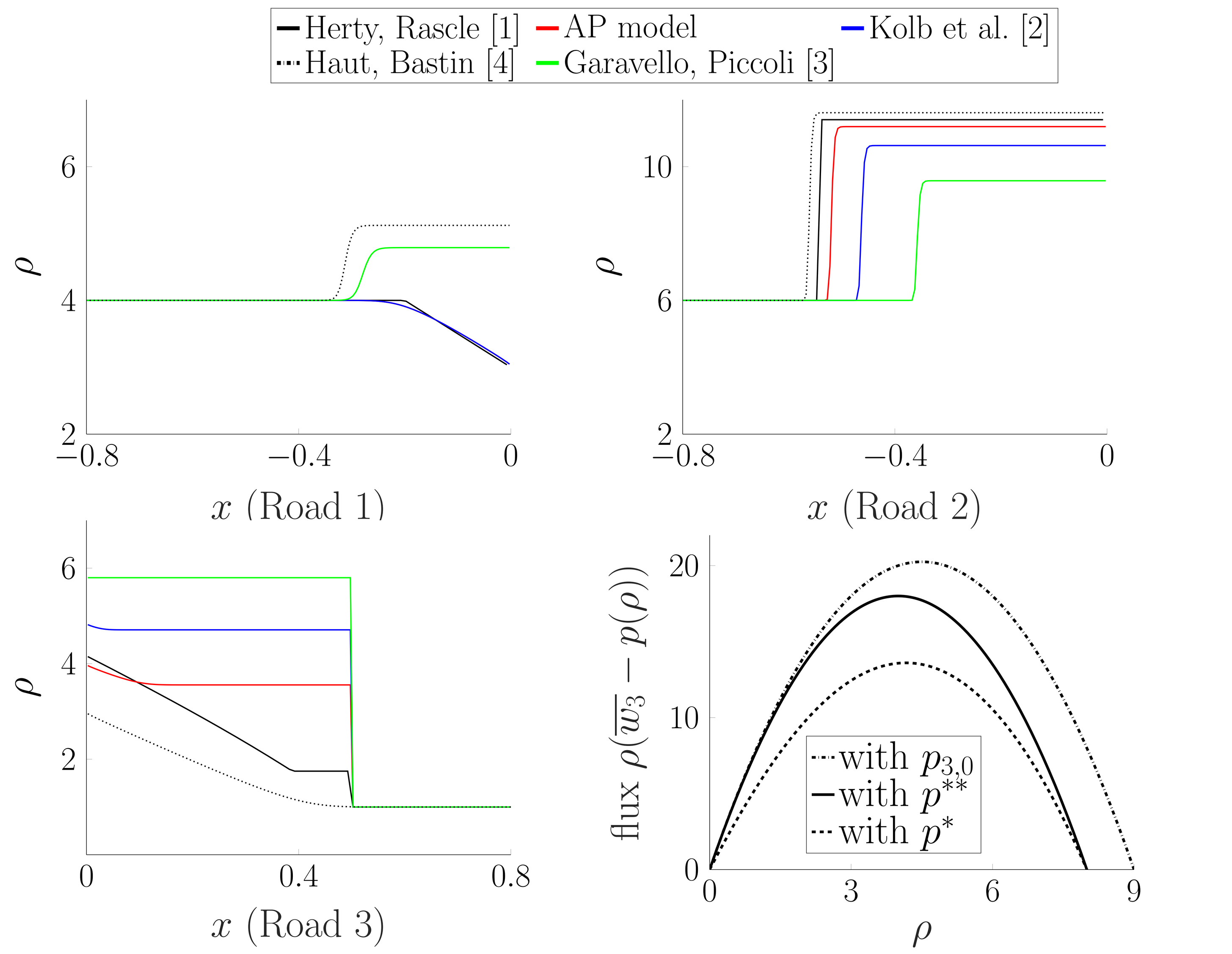
The merging junction



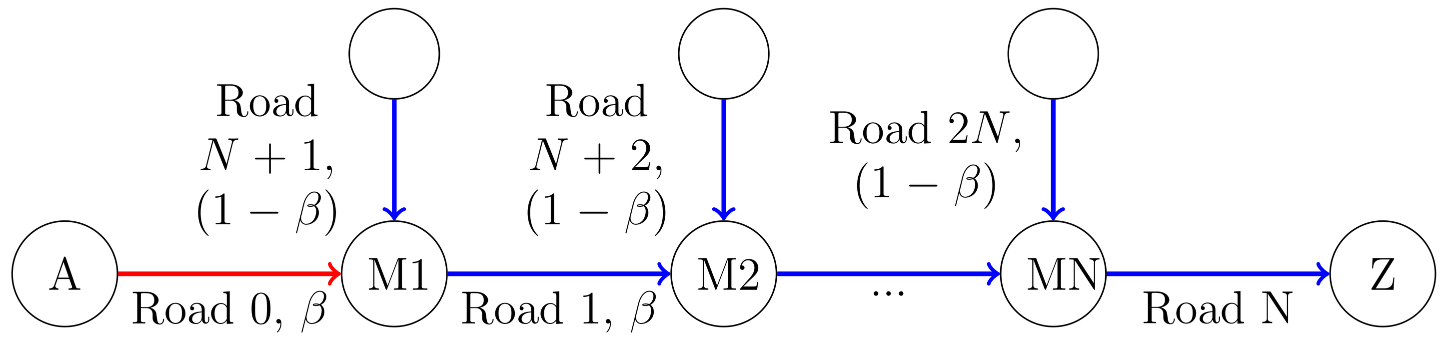
Consider a merge of two incoming roads $i = 1, 2$ into a single outgoing road $j = 3$. Let $\vec{\beta} = (\beta_1, \beta_2) = (0.5, 0.5)$, $(\rho_{1,0}, \rho_{2,0}, \rho_{3,0}) = (4, 6, 4)$, $(w_{1,0}, w_{2,0}, w_{3,0}) = (6, 12, 6)$ and $p_{i,0}(\rho) = \rho, i = 1, 2, 3$, then

$$\overline{w_3} = \beta_1 w_{1,0} + \beta_2 w_{2,0} = 9, \qquad p_3^{**}(\rho) = c_{3,0}\left(1 + \beta(1-\beta)\frac{(w_{1,0} - w_{2,0})^2}{w_{1,0}w_{2,0}}\right)\rho = 1.125\rho.$$

The figures below compare our network solution to other network solutions ($t = 0.1$).



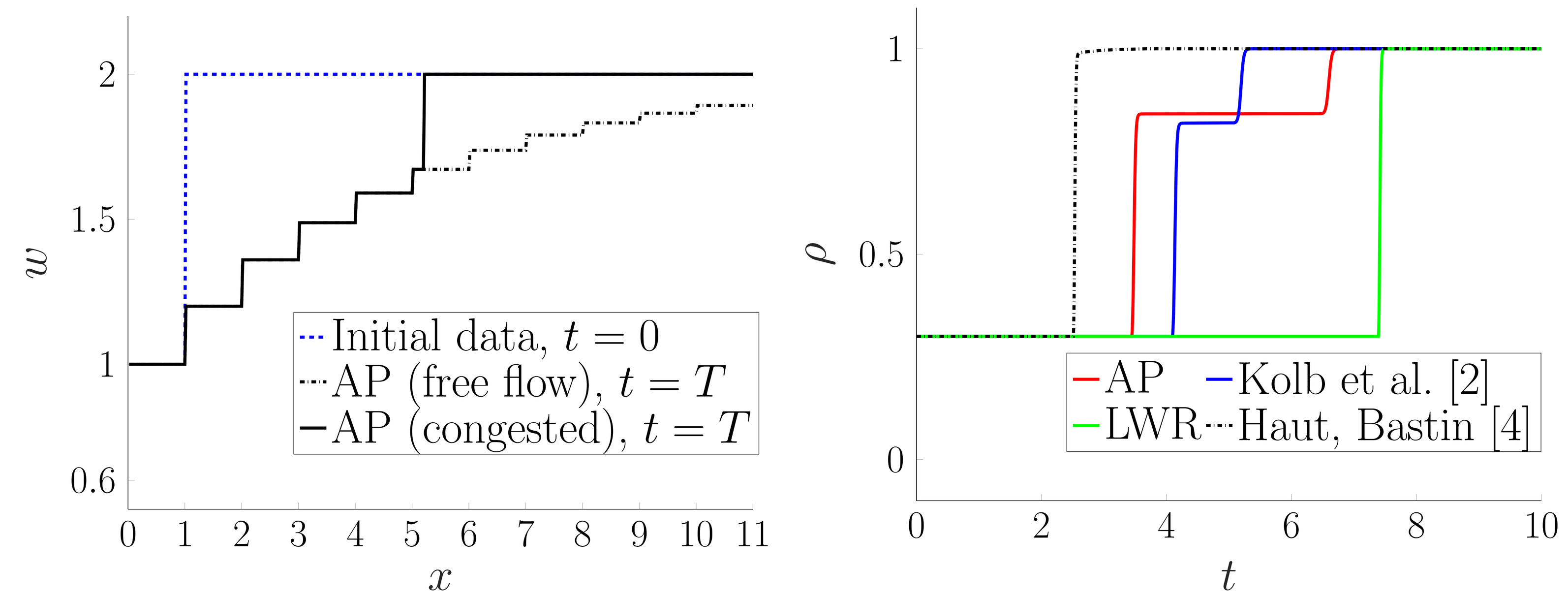
A sequential network of merging junctions



We set $w_{0,0} = 1$, $w_{i,0} = 2, i > 0$, $\beta = 0.8, p_{i,0} = \rho$, $N = 10$. The length of each road is $L_i = 0.5$ and the end of the time horizon is $T = 10$. We consider two scenarios with $\rho_{i,0} = 0.3$, $i = 1, \ldots, N-1, N+1, \ldots 2N$

<u>free flow scenario:</u> $\rho_{N,0} = 0.3$ <u>congested scenario:</u> $\rho_{N,0} = 1$.

The Lagrangian marker on the main roads 0-10 at $t = T$ for the free flow and the congested scenario is depicted in the left figure. In the congested scenario, the last road is fully congested and a shock wave moves backwards through the network. The right figure shows the density at the beginning of Road 0 over time in the congested scenario and illustrates the propagation of the traffic jam from the back to the front of the network.



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The Aw-Rascle-Zhang model

On each road i of the network, we have

$$\partial_t \begin{pmatrix} \rho_i \\ \rho_i w_i \end{pmatrix} + \partial_x \begin{pmatrix} \rho_i v_i \\ \rho_i w_i v_i \end{pmatrix} = 0,$$
$$w_i = v_i + p_i(\rho_i)$$

where ρ_i density, v_i velocity, $p_i(\rho)$ pressure, w_i Lagrangian marker

New approach:

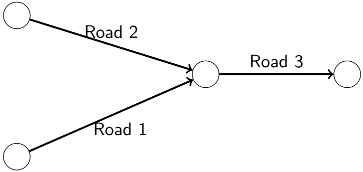
- novel approximation to the **homogenized pressure**
- **additional equation** for the propagation of a reference pressure
- starting from the **junctions of the network**

Adapted pressure (AP) model

$$\partial_t \begin{pmatrix} \rho \\ \rho w \\ \rho c \end{pmatrix} + \partial_x \begin{pmatrix} \rho v \\ \rho w v \\ \rho c v \end{pmatrix} = 0.$$

with eigenvalues
$\lambda_1 = v - p'(\rho)\rho,$
$\lambda_2 = \lambda_3 = v.$

- c might change due to the coupling at the junction
- pressure law is of the type $p(\rho) = c(\vec{\beta}, \vec{w}) p_{j,0}(\rho)$
- $\vec{\beta}$ priority vector, $\vec{w}$ Lagrangian marker



The merging junction

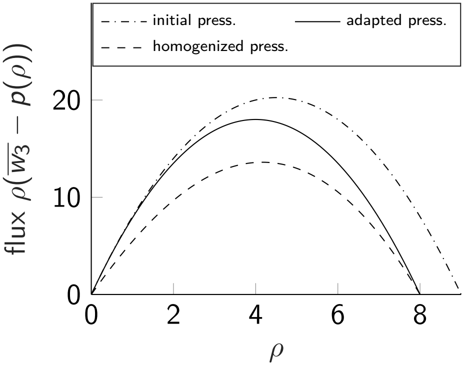
Fixed ratios $\beta_1 = \beta$ and $\beta_2 = 1 - \beta$ are assigned to the incoming roads. Initially, $p_{3,0} = \rho$.

Lagrangian marker

$$\overline{w_3} = \beta_1 w_{1,0} + \beta_2 w_{2,0}$$

Adapted pressure law (AP)

$$p_3^{**}(\rho) = c_{3,0} \left(\sum_{i=1}^{2} \sum_{l=1}^{2} \frac{\beta_i \beta_l w_{l,0}}{w_{i,0}} \right) \rho$$



The Riemann Problem

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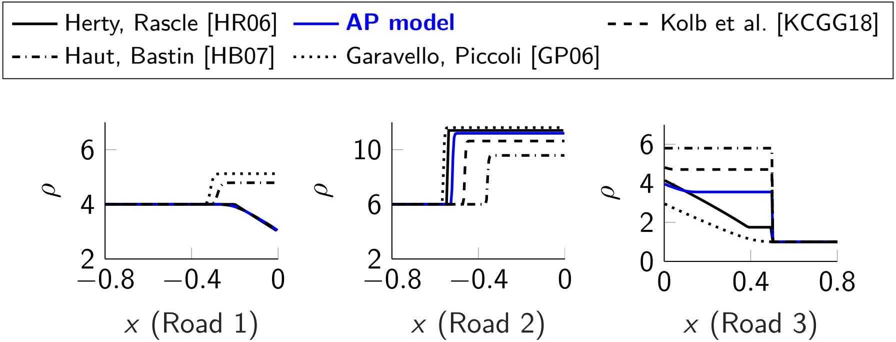
For $\vec{\beta} = (0.5, 0.5)$, $(\rho_{1,0}, \rho_{2,0}, \rho_{3,0}) = (4, 6, 4)$, $(w_{1,0}, w_{2,0}, w_{3,0}) = (6, 12, 6)$ and $p_{i,0}(\rho) = \rho$, $i = 1, 2, 3$, we obtain $p_3^{**}(\rho) = 1.125\rho$.



A sequential network of merging junctions

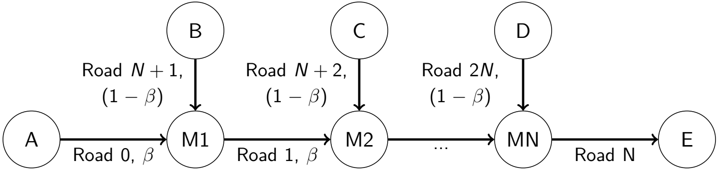


Figure 1: Network of merging junctions.

A sequential network of merging junctions

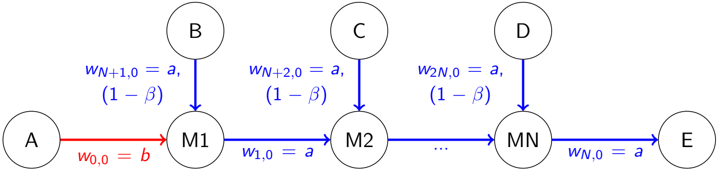
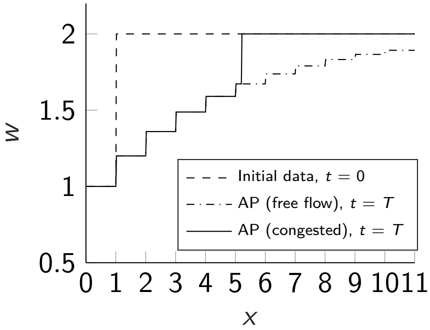


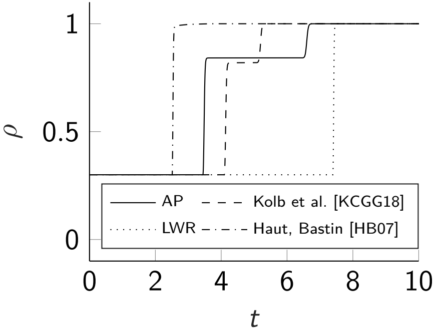
Figure 2: Network of merging junctions.

Free flow and congested scenario

We set $w_{0,0} = 1$, $w_{i,0} = 2, i > 0$, $\beta = 0.8, p_{i,0} = \rho$, $N = 10$. The length of each road is $L_i = 0.5$ and the end of the time horizon is $T = 10$.
free flow scenario: $\rho_{i,0} = 0.3, i = 1, \ldots, 2N$

congested scenario: $\rho_{N,0} = 1$ and $\rho_{i,0} = 0.3, i = 1, \ldots, N-1, N+1, \ldots 2N$





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