Isoperimetric Inequalities for Polar L_p Centroid Bodies

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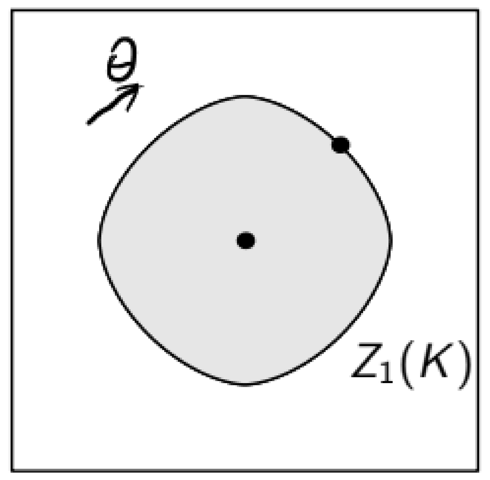
L_p Centroid Bodies

Centroid Bodies (Geometric Definition): Given an origin-symmetric convex body K in $\mathbb{R}^n$ and $\theta \in S^{n-1}$, $\theta^\perp$ divides K into two symmetric halves. The boundary of $Z_1(K)$ is the collection of the centroids of the halves

$$K \cap \theta^\perp_+ = \{x \in K \mid \langle x, \theta\rangle \geq 0\}. \quad (1)$$

Fundamental isoperimetric inequalities for such bodies were proved by Blaschke, Busemann, and Petty (see [Gar06]).



L_p Centroid Bodies (Analytic Definition): Let K be a convex body, $K = -K$, in $\mathbb{R}^n$ and $p > 0$, define $Z_p(K)$ by its support function

$$h_{Z_p(K)}(\theta) = \left(\frac{1}{|K|}\int_K |\langle x, \theta\rangle|^p \, dx\right)^{\frac{1}{p}}. \qquad (\theta \in S^{n-1}) \quad (2)$$

Isoperimetric Inequalities for L_p Centroid Bodies, $p \geq 1$: First established by Lutwak and Zhang [LZ97]:

Theorem. *Let K be a star body. For $1 \leq p < \infty$,*

$$|Z_p^\circ(K)| \leq |Z_p^\circ(K^*)|, \quad (3)$$

where $Z_p^\circ(K)$ denotes the polar of $Z_p(K)$ and K^* is the dilate of the ball in $\mathbb{R}^n$ so that $|K| = |K^*|$. (3) is an equality iff K is an origin-symmetric ellipsoid.

Later this was extended to $Z_p(K)$ itself.

Theorem (Lutwak-Yang-Zhang [LYZ00]). *With the same assumption,*

$$|Z_p(K)| \geq |Z_p(K^*)|, \quad (4)$$

with equality iff K is an origin-symmetric ellipsoid.

Proofs of both theorems utilize Steiner symmetrization.

Empirical L_p Centroid Bodies

L_p Centroid Body (Probabilistic Approach): Let $X_1, \cdots, X_N$ be iid random vectors in $\mathbb{R}^n$ drawn from a continuous distribution f. For $p \geq 1$, define the empirical L_p centroid body via

$$h_{Z_{p,N}(f)}(\theta) = \left(\frac{1}{N}\sum_{i=1}^N |\langle X_i, \theta\rangle|^p\right)^{\frac{1}{p}} \qquad (\theta \in S^{n-1}) \quad (5)$$

By the Law of Large Numbers, $h_{Z_{p,N}(f)}$ converges to $h_{Z_p(f)}$ almost surely (where $Z_p(f)$ is defined as in (2) by replacing $\chi_K/|K|$ with f). Alternatively,

$$Z_{p,N}(f) = N^{-\frac{1}{p}} X B_q^N \quad (6)$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $X = \begin{bmatrix} X_1 & \cdots & X_N \end{bmatrix}$. Paouris and Pivovarov [PP12] proved

$$\mathbb{E}|Z_{p,N}(f)| \geq \mathbb{E}|Z_{p,N}(f^*)| \quad (7)$$

where f^* is the symmetric decreasing rearrangement of f.

Beyond the $p \geq 1$ Case: The radial function of the polar empirical L_p centroid body for $0 < p < 1$ is given by

$$\rho_{Z^\diamond_{p,N}(f)}(\theta) = \left(\frac{1}{N}\sum_{i=1}^N |\langle X_i, \theta\rangle|^p\right)^{-\frac{1}{p}} \qquad (\theta \in S^{n-1}) \quad (8)$$

As N tends to infinity, $Z^\diamond_{p,N}(f)$ approaches the deterministic polar L_p centroid body as defined by[YY06], which is given by

$$\rho_{Z^\diamond_p(f)}(\theta) = \left(\int_{\mathbb{R}^n} |\langle x, \theta\rangle|^p f(x)\, dx\right)^{-\frac{1}{p}} \qquad (\theta \in S^{n-1}) \quad (9)$$

Such bodies interpolate between intersection and L_p centroid bodies [HL06, Kol05]. Berck [Ber09] showed that $Z_p(f)$ is convex when $f = \chi_K$, K an origin-symmetric convex body. However $Z^\diamond_{p,N}(f)$ is only a star body, so new tools not relying on convexity are needed.

Main Result:

Theorem. *Let f be as above and $0 < p < 1$. Then*

$$\mathbb{E}|Z^\diamond_{p,N}(f)| \leq \mathbb{E}|Z^\diamond_{p,N}(f^*)|. \quad (10)$$

Consequently as $N \to \infty$

$$|Z^\diamond_p(f)| \leq |Z^\diamond_p(f^*)| \quad (11)$$

Sketch of Proof

A Volume Formula: The main technical ingredient is a volume formula by Nayar and Tkocz [NT20]. A random variable W is α-stable with density g_α for $0 < \alpha < 1$ if

$$\mathbb{E}e^{-tW} = \mathbb{E}e^{-t^\alpha} \quad (12)$$

Theorem. *Let $0 < p < 2$ and X be an $n \times N$ matrix with columns $X_1, \cdots, X_N$. Let $W_1, \cdots, W_N$ be iid random variables with density $t^{-1/2} g_{p/2}(t)$. Then*

$$|B_p^N \cap \mathrm{Im}\, X^*| = c_{n,p}|XX^*|^{-\frac{1}{2}} \mathbb{E}_W \prod_{i=1}^N W_i^{-\frac{1}{2}} \left|\sum_{i=1}^N W_i X_i X_i^*\right|^{-\frac{1}{2}} \quad (13)$$

A Rearrangement Inequality: Call $F : (\mathbb{R}^n)^N \to \mathbb{R}$ Steiner concave if for any fixed $\theta \in \mathbb{R}^n$ and $(y_1, \cdots, y_N) \in (\theta^\perp)^N$ the function

$$F_Y(t_1, \cdots, t_N) = F(y_1 + t_1\theta, \cdots, y_N + t_N\theta) \quad (14)$$

is even and quasi-concave (i.e. its level sets are convex).

Theorem (Brascamp-Lieb-Luttinger [Chr84]). *If $F : (\mathbb{R}^n)^N \to \mathbb{R}$ is Steiner concave and $f_i : \mathbb{R}^n \to \mathbb{R}$, $1 \leq 1 \leq N$, is of class $L^1(\mathbb{R}^n)$, then*

$$\int_{(\mathbb{R}^n)^N} F(\overline{x}) \prod_{i=1}^N f_i(x_i)\, d\overline{x} \leq \int_{(\mathbb{R}^n)^N} F(\overline{x}) \prod_{i=1}^N f_i^*(x_i)\, d\overline{x} \quad (15)$$

Key Observation: Reinterpreting (6) and combining with (13) gives

$$|Z^\diamond_{p,N}(f)| = c_{N,n,p}\mathbb{E}_W \prod_{i=1}^N W_i^{\frac{1}{2}} \left|X(\sqrt{W}B_2^N)\right|^{-1} \quad (16)$$

where $\sqrt{W} = \mathrm{diag}(W_1, \cdots, W_N)$.
Complete with the following theorem from [PP12]

Theorem. *Let X be as above and C a convex body in $\mathbb{R}^N$. If $F : (\mathbb{R}^n)^N \to R$ is defined by*

$$F(X_1, \cdots, X_N) = |XC|^{-1}, \quad (17)$$

then F is Steiner concave.

Alternative Approach: Use a more general stochastic inequality: from [CEFPP15]

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