Minimal Surface Generating Flow

Abstract

This work introduces new geometric flow for space curves with positive curvature and torsion. Curves evolving according to this motion law trace out a zero mean curvature surface. We present properties of the motion law, discuss its limitations and sketch future work directions.

Geometric Flow of Space Curves

Let $\{\Gamma_t\}_{t\in[0,t_{max})}$ be a family of curves in $\mathbb{R}^3$ evolving in time interval $[0, t_{max})$. Each curve Γ_t is described by a parametric function

$$X(\cdot, t) : \mathbb{S}^1 \to \mathbb{R}^3.$$

Assume X is differentiable, $X(\cdot, t)$ is $C^2(\mathbb{S}^1; \mathbb{R}^3)$ for all $t \in [0, t_{max})$ and $\|\partial_u X\| > 0$ everywhere. General geometric flow is given by the following initial-value problem:

$$\begin{aligned} \partial_t X &= \beta N + \gamma B && \text{in } \mathbb{S}^1 \times (0, t_{max}), \\ X|_{t=0} &= X_0 && \text{in } \mathbb{S}^1, \end{aligned}$$

where X_0 is the parametrization for the initial curve Γ_0 and N, B are the principal normal and binormal vector, respectively. The velocity functions $\beta(u, t)$ and $\gamma(u, t)$ depend only on local quantities of Γ_t at the point $X(u, t)$.

Trajectory Surfaces

For given velocities β and γ, terminal time t_{max} and an initial curve Γ_0, we formally define the **trajectory surface** $\Sigma_{t_{max}}$ as

$$\Sigma_{t_{max}} := \bigcup_{t\in[0,t_{max})} \Gamma_t.$$

The Gaussian curvature K and mean curvature H of $\Sigma_{t_{max}}$ are

$$\begin{aligned} K =& \kappa\gamma \frac{\gamma\partial_t\beta - \beta\partial_t\gamma}{(\beta^2+\gamma^2)^2} - \gamma\frac{\beta\partial_s\tau + 2\tau\partial_s\beta + \partial_s^2\gamma - \gamma\tau^2}{(\beta^2+\gamma^2)^{\frac{3}{2}}} \\ & - \frac{(\beta\partial_s\gamma - \gamma\partial_s\beta)^2}{(\beta^2+\gamma^2)^2} - 2\tau\frac{\beta\partial_s\gamma - \gamma\partial_s\beta}{\beta^2+\gamma^2} - \tau^2, \\ H =& -\frac{\kappa\gamma}{\sqrt{\beta^2+\gamma^2}} + \frac{\beta\partial_t\gamma - \gamma\partial_t\beta}{(\beta^2+\gamma^2)^{\frac{3}{2}}} + \frac{\beta\partial_s\tau + 2\tau\partial_s\beta + \partial_s^2\gamma - \gamma\tau^2}{\kappa\sqrt{\beta^2+\gamma^2}}, \end{aligned}$$

where κ, τ is the curvature and the torsion of Γ_t.

Minimal Surface Generating Flow

Assuming $\gamma = 0$, we get

$$H = \frac{\beta\partial_s\tau + 2\tau\partial_s\beta}{|\beta|\kappa} = \frac{\beta\tau}{|\beta|\kappa}\partial_s\left(\log|\tau| + 2\log|\beta|\right).$$

The mean curvature H is zero everywhere, if and only if $\beta = F\tau^{-\frac{1}{2}}$, where $F = F(t)$ is a function independent of the parameter u. Thus we arrive at the **Minimal Surface Generating flow** [2]:

$$\begin{aligned} \boxed{\partial_t X = \tau^{-\frac{1}{2}} N} && \text{in } \mathbb{S}^1 \times (0, t_{max}), \\ X|_{t=0} = X_0 && \text{in } \mathbb{S}^1, \end{aligned}$$

where X_0 is parametrization of the initial curve Γ_0.

Properties of $\Sigma_{t_{max}}$

- Zero mean curvature surface ($H \equiv 0$).
- Topological annulus with boundaries Γ_0 and $\Gamma_{t_{max}}$.
- Principal curvatures at $X(u, t)$ are $\kappa_{1,2} = \pm\tau(u, t)$.
- It is contained in the convex hull of Γ_0 (Maximum Principle).

Generated Surface Area

Using Gauss-Bonnet formula

$$\int_{\Sigma_{t_{max}}} K \,\mathrm{d}A + \int_{\partial\Sigma_{t_{max}}} \kappa_g \,\mathrm{d}s = 2\pi\chi(\Sigma_{t_{max}}),$$

which in our case translates to

$$-\int_0^{t_{max}} \left(\int_{\Gamma_t} \tau^{\frac{3}{2}} \,\mathrm{d}s\right) \mathrm{d}t + \int_{\Gamma_0} \kappa \,\mathrm{d}s - \int_{\Gamma_{t_{max}}} \kappa \,\mathrm{d}s = 0,$$

allows us to prove the following bound for the total **area swept by the moving curve**:

$$A(\Sigma_{t_{max}}) < \frac{1}{\inf_{\Gamma_0} \tau^2}\left(\int_{\Gamma_0} \kappa \,\mathrm{d}s - 2\pi\right).$$

The constant 2π comes from the **Fenchel's Theorem**. If the initial curve is knotted and stays knotted during the evolution, one can replace it by 4π using the **Fáry–Milnor Theorem**.

Global Quantities

The total square root of torsion is preserved, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_t} \tau^{\frac{1}{2}} \,\mathrm{d}s = 0.$$

The length $L(\Gamma_t)$ monotonically decreases, because

$$\frac{\mathrm{d}}{\mathrm{d}t} L(\Gamma_t) = -\int_{\Gamma_t} \kappa\tau^{-\frac{1}{2}} \,\mathrm{d}s < 0.$$

If $t_{max} = +\infty$, then **the curve shrinks to a point**, i.e. $\lim_{t\to+\infty} L(\Gamma_t) = 0$.

Terminal Time

The evolution equations for curvature κ and torsion τ read

$$\begin{aligned} \partial_t\kappa &= \partial_s^2(\tau^{-\frac{1}{2}}) + \tau^{-\frac{1}{2}}(\kappa^2 - \tau^2), \\ \partial_t\tau &= 2\kappa\tau^{\frac{1}{2}}. \end{aligned}$$

By analyzing the evolution of **averaged curvature** $\langle\kappa(\cdot, t)\rangle$, defined as

$$\langle\kappa(\cdot, t)\rangle := \frac{1}{L(\Gamma_t)}\int_{\Gamma_t} \kappa(s, t) \,\mathrm{d}s,$$

we can bound the terminal time t_{max} as

$$t_{max} \le \frac{1}{2\inf_{\Gamma_0} \tau^{\frac{1}{2}}} \log\left(1 + \frac{2\langle\kappa(\cdot, 0)\rangle}{\inf_{\Gamma_0} \tau - \langle\kappa(\cdot, 0)\rangle}\right).$$

for all initial curves Γ_0 satisfies $\inf_{\Gamma_0} \tau < \langle\kappa(\cdot, 0)\rangle$.

Example of Analytical Solution

Consider the **helix curve**, given by

$$X_0(u) = (r_0\cos(u), r_0\sin(u), cu)^T,$$

for $u \in \mathbb{R}$ with $r_0, c > 0$.

Given the symmetries of helix, we can look for the solution in the form

$$X(u, t) = (r(t)\cos(u), r(t)\sin(u), cu)^T.$$

Since $\tau = c(r^2 + c^2)^{-1}$, the problem reduces to an ODE

$$\dot{r}(t) = -(c^{-1}r(t)^2 + c)^{\frac{1}{2}}.$$

The solution monotonically decreases its radius and converges to a line, leaving behind the **helicoid minimal surface**.

Limitations

- The flow is restricted to a class of locally convex curves with positive torsion. Consequences of this restriction were further analyzed in [3].
- The surface $\Sigma_{t_{max}}$ cannot become a disc in the limit if the Frenet frame of Γ_0 is not a Seifert framing.
- Despite promising results suggesting long-term existence of the solution, we were not able to obtain any local existence result yet.

Future Work

To avoid some of the limitations mentioned above, we are currently working on the following formulation:

$$\begin{aligned} \partial_t X &= \kappa\cos\theta N + \kappa\sin\theta B && \text{in } \mathbb{S}^1 \times (0, t_{max}), \\ X|_{t=0} &= X_0 && \text{in } \mathbb{S}^1, \end{aligned}$$

where the angle function $\theta\colon \mathbb{S}^1 \times [0, t_{max}) \mapsto \mathbb{R}$ is defined below. This motion law is a combination of the curve shortening flow in $\mathbb{R}^3$ (see [1]) and binormal flow. This motion law generates trajectory surface with mean curvature

$$\begin{aligned} H = & \tfrac{1}{\kappa}\partial_t\theta + \tfrac{1}{\kappa^2}\cos\theta\left[\kappa\partial_s\tau + 2\tau\partial_s\kappa + 2\partial_s\kappa\partial_s\theta + \kappa\partial_s^2\theta\right] \\ & + \tfrac{1}{\kappa^2}\sin\theta\left[\partial_s^2\kappa - \kappa^3 - 2\tau\kappa\partial_s\theta - \kappa(\partial_s\theta)^2 - \kappa\tau^2\right]. \end{aligned}$$

Letting $H = 0$, gives us

$$\begin{aligned} \partial_t\theta = & -\partial_s^2\theta\cos\theta - (\partial_s\theta)^2\sin\theta - 2\partial_s\theta\left(\kappa^{-1}\partial_s\kappa\cos\theta - \tau\sin\theta\right) \\ & - \partial_s\tau\cos\theta - 2\tau\kappa^{-1}\partial_s\kappa\cos\theta - \kappa^{-1}\partial_s^2\kappa\sin\theta + \kappa^2\sin\theta + \tau^2\sin\theta. \end{aligned}$$

References

[1] J. Minarčík, M. Beneš. *Long-term behavior of curve shortening flow in* $\mathbb{R}^3$ SIAM Journal on Mathematical Analysis, **52** (2020), 1221–1231.

[2] J. Minarčík, M. Beneš. *Minimal surface generating flow for space curves of non-vanishing torsion.* Discrete & Continuous Dynamical Systems - B, to appear (2022).

[3] J. Minarčík, M. Beneš. *Nondegenerate Homotopy and Geometric Flows.* Homology, Homotopy and Applications, to appear (2022).

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