Hermitian Tensor Decompositions

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Introduction

A tensor $\mathcal{H} = (\mathcal{H}_{i_1 \dots i_m j_1 \dots j_m}) \in \mathbb{C}^{n_1 \times \cdots \times n_m \times n_1 \times \cdots \times n_m}$ is **Hermitian** if

$$\mathcal{H}_{i_1 \dots i_m j_1 \dots j_m} = \overline{\mathcal{H}_{j_1 \dots j_m i_1 \dots i_m}}.$$

- The set of all such Hermitian tensors is denoted by $\mathbb{C}^{[n_1,\dots,n_m]}$.
- Mixed quantum stated can be represented by Hermitian tensors.
- A rank-1 Hermitian tensor must have form

$$[v^1, v^2, \cdots, v^m]_{\otimes h} := v^1 \otimes v^2 \cdots \otimes v^m \otimes \overline{v^1} \otimes \overline{v^2} \cdots \otimes \overline{v^m}.$$

- Every Hermitian tensor is a sum of rank-1 Hermitian tensors. The smallest such length is called **Hermitian rank** of $\mathcal{H}$, denoted by hrank$(\mathcal{H})$.

The following topics are studied in the work:

- The Hermitian rank of basis Hermitian tensors.
- Real Hermitian tensors
- Matrix flattenings of Hermitian tensors.
- Positivity and separability of Hermitian Tensors
- How to detect separability.

Basis Hermitian tensors

Let

$$I = (i_1, \dots, i_m), \quad J = (j_1, \dots, j_m),$$

For a complex number c, $\mathcal{E}^{IJ}(c)$ is the basis tensor such that

$$[\mathcal{E}^{IJ}(c)]_{IJ} = \overline{[\mathcal{E}^{IJ}(c)]_{JI}} = c$$

with other entries being 0.

Theorem 1. *Assume $n_1, \dots, n_m \geq 2$ and $c \neq 0$. If $I = J$, then hrank$(\mathcal{E}^{IJ}(c)) = 1$; if $I \neq J$, then hrank$(\mathcal{E}^{IJ}(c)) = 2d$ where d is the number of nonzero entries of $I - J$.*

For $n = (2, 2, \dots, 2)$ and $I = (1, 1, \dots, 1)$, $J = (2, 2, \dots, 2)$, the basis tensor $\mathcal{E}^{IJ}(c)$ has the decomposition

$$\mathcal{E}^{IJ}(c) = \frac{1}{2m}\Big([\tilde{u}_0, u_0, \dots, u_0]_{\otimes h} + (-1)^m [\tilde{u}_m, u_m, \dots, u_m]_{\otimes h} + \sum_{k=1}^{m-1} (-1)^k ([\tilde{u}_k, u_k, \dots, u_k]_{\otimes h} + [\tilde{v}_k, \overline{u_k}, \dots, \overline{u_k}]_{\otimes h})\Big),$$

where $u_k := (1, \exp(\frac{k}{m}\pi\sqrt{-1}))$, $\tilde{u}_k = (c, \exp(\frac{k}{m}\pi\sqrt{-1}))$ and $\tilde{v}_k = (c, \exp(-\frac{k}{m}\pi\sqrt{-1}))$.

Example 2. For $I = (1, 2)$, $J = (3, 4)$ and $c \neq 0$, the basis tensor $\mathcal{E}^{(12)(34)}(c) \in \mathbb{C}^{[4,4]}$ has the Hermitian rank 4, with the following Hermitian rank decomposition (in the following $i := \sqrt{-1}$)

$$\frac{1}{4}\left[\begin{pmatrix} c \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}\right]_{\otimes h} + \frac{1}{4}\left[\begin{pmatrix} c \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}\right]_{\otimes h} - \frac{1}{4}\left[\begin{pmatrix} c \\ 0 \\ i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ i \end{pmatrix}\right]_{\otimes h} - \frac{1}{4}\left[\begin{pmatrix} c \\ 0 \\ -i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -i \end{pmatrix}\right]_{\otimes h}.$$

Real Hermitian tensors

Real Hermitian tensors are Hermitian tensors whose entries are all real. The set of all real Hermitian tensors is denoted by $\mathbb{R}^{[n_1,\cdots,n_m]}$.
A tensor $\mathcal{H} \in \mathbb{R}^{[n_1,\cdots,n_m]}$ is called $\mathbb{R}$-*Hermitian decomposable* if

$$\mathcal{H} = \sum_{i=1}^{r} \lambda_i [u_i^1, \dots, u_i^m]_{\otimes h} \tag{1}$$

where $\lambda_i \in \mathbb{R}$ and $u_i^k \in \mathbb{R}^{n_k}$. The set of all $\mathbb{R}$-Hermitian decomposable tensors is denoted by $\mathbb{R}_D^{[n_1,\cdots,n_m]}$.

Not every real Hermitian tensor is real decomposable. Let $\mathbb{R}_D^{[n_1,\cdots,n_m]}$ be the space of all real Hermitian decomposable tensors,then

$$\dim \mathbb{R}_D^{[n_1,\cdots,n_m]} = \prod_{k=1}^{m} \binom{n_k + 1}{2} = \prod_{k=1}^{m} \frac{n_k(n_k+1)}{2}. \tag{2}$$

However, the dimension of all real Hermitian tensors is

$$\dim \mathbb{R}^{[n_1,\cdots,n_m]} = \binom{n_1 \cdots n_m + 1}{2}. \tag{3}$$

Thus if $m > 1, n_i > 1$, then $\dim \mathbb{R}^{[n_1,\cdots,n_m]} > \dim \mathbb{R}_D^{[n_1,\cdots,n_m]}$.

Theorem 3. A tensor $\mathcal{A} \in \mathbb{R}^{[n_1,\cdots,n_m]}$ is real Hermitian decomposable if and only if

$$\mathcal{A}_{i_1 \dots i_m j_1 \dots j_m} = \mathcal{A}_{k_1 \dots k_m l_1 \dots l_m} \tag{4}$$

for all labels such that $\{i_s, j_s\} = \{k_s, l_s\}$ for all $s = 1, \dots, m$.

Matrix flattenings

For the Hermitian tensor $\mathcal{H}$ with the decomposition $\mathcal{H} := \sum_{i=1}^r \lambda_i [u_i^1, \dots, u_i^m]_{\otimes h}$, its Hermitian flattening matrix is

$$\mathfrak{m}(\mathcal{H}) = \sum_{i=1}^{r} \lambda_i \, (u_i^1 \boxtimes \cdots \boxtimes u_i^m)(u_i^1 \boxtimes \cdots \boxtimes u_i^m)^*,$$

where $\boxtimes$ is the Kronecker product and a^* is the conjugate transpose of a. The canonical Kronecker flattening of $\mathcal{H}$ is

$$\kappa(\mathcal{H}) := \sum_{i=1}^{r} (u_i^1 \boxtimes \overline{u_i^1} \boxtimes \cdots \boxtimes u_i^{m-1} \boxtimes \overline{u_i^{m-1}})(u_i^m \boxtimes \overline{u_i^m})^T.$$

rank$(\mathfrak{m}(\mathcal{H}))$ and rank$(\kappa(\mathcal{H}))$ are both lower bounds for hrank$(\mathcal{H})$. However, the bounds can be very different.

Example 4. For $m = 2$ and $n > 1$, consider the Hermitian tensor in $\mathbb{R}^{[n,n]}$

$$\mathcal{H} = \sum_{i,j=1}^{n} e_i \otimes e_i \otimes e_j \otimes e_j = \big(\sum_{i=1}^{n} e_i \otimes e_i\big) \otimes \big(\sum_{i=1}^{n} e_i \otimes e_i\big).$$

Then,

$$\mathfrak{m}(\mathcal{H}) = \big(\sum_{i=1}^{n} e_i \boxtimes e_i\big)\big(\sum_{i=1}^{n} e_i \boxtimes e_i\big)^T, \ \kappa_\phi(\mathcal{H}) = \big(\sum_{i=1}^{n} e_i e_i^T\big) \boxtimes \big(\sum_{i=1}^{n} e_i e_i^T\big) = I_{n^2}.$$

Thus, it holds that hrank$(\mathcal{H}) \geq$ rank$\kappa(\mathcal{H}) = n^2$ while rank $\mathfrak{m}(\mathcal{H}) = 1$.

Positivity and separability

- A Hermitian tensor $\mathcal{H} \in \mathbb{H}^{[n_1,\dots,n_m]}$ can be uniquely determined by the conjugate multi-quadratic polynomial

$$\mathcal{H}(x, \overline{x}) := \sum_{i_1,\dots,i_m,j_1,\dots,j_m} \mathcal{H}_{i_1 \dots i_m j_1 \dots j_m} (x_1)_{i_1} \cdots (x_m)_{i_m} (\bar{x}_1)_{i_1} \cdots (\bar{x}_m)_{i_m}$$

  in the tuple $x := (x_1, \dots, x_m)$ of complex vector variables $x_i \in \mathbb{C}^{n_i}$.
- $\mathcal{H} \in \mathcal{F}^{[n_1,\dots,n_m]}$ ($\mathcal{F} = \mathbb{C}$ or $\mathbb{R}$) is called $\mathcal{F}$-psd if $\mathcal{H}(x, \overline{x}) \geq 0, \forall x_i \in \mathcal{F}^{n_i}$. Denote the cone of $\mathcal{F}$-psd Hermitian tensors by $\mathscr{P}_{\mathcal{F}}^{[n_1,\dots,n_m]}$.
- $\mathcal{H} \in \mathcal{F}^{[n_1,\dots,n_m]}$ is called $\mathcal{F}$-*separable*if $\mathcal{H}$ has the decomposition $\mathcal{H} = \sum_{i=1}^r [u_i^1, \dots, u_i^m]_{\otimes h}$ for some $u_i^j \in \mathcal{F}^{n_j}$. Denote the cone of $\mathcal{F}$-separable Hermitian tensors by $\mathscr{S}_{\mathcal{F}}^{[n_1,\dots,n_m]}$.
- The separability of Hermitian tensors are equivalent to the separability of mixed quantum states.

The following theorem characterizes properties of $\mathscr{S}_{\mathcal{F}}^{[n_1,\dots,n_m]}$, $\mathscr{P}_{\mathcal{F}}^{[n_1,\dots,n_m]}$ and their duality relationship.

Theorem 5. *$\mathscr{P}_{\mathbb{C}}^{[n_1,\dots,n_m]}$ and $\mathscr{S}_{\mathbb{C}}^{[n_1,\dots,n_m]}$ are proper cones, i.e. they are closed, convex, solid, and pointed. $\mathscr{P}_{\mathbb{R}}^{[n_1,\dots,n_m]}$ and $\mathscr{S}_{\mathbb{R}}^{[n_1,\dots,n_m]}$ are closed and convex. However, $\mathscr{P}_{\mathbb{R}}^{[n_1,\dots,n_m]}$ is solid but not pointed; $\mathscr{S}_{\mathbb{R}}^{[n_1,\dots,n_m]}$ is pointed but not solid. Moreover, $\mathscr{S}_{\mathcal{F}}^{[n_1,\dots,n_m]}$ and $\mathscr{P}_{\mathcal{F}}^{[n_1,\dots,n_m]}$ are dual to each other for $\mathcal{F} = \mathbb{R}, \mathbb{C}$.*

Detect separability

Theorem 6. *The tensor $\mathcal{H} \in \mathcal{F}^{[n_1,\dots,n_m]}$ is $\mathcal{F}$-separable if and only if there exists a Borel measure μ such that*

$$\mathcal{H} = \int [x_1, \dots, x_m]_{\otimes h} d\mu.$$

Checking separability is equivalent to checking the following moment problem.

$$\begin{array}{rl} \min_\mu & \int F(\tilde{x}) d\mu \\ \text{s.t.} & \mathrm{Re}(\mathcal{H}) = \int R(\tilde{x}) d\mu \\ & \mathrm{Im}(\mathcal{H}) = \int I(\tilde{x}) d\mu, \end{array} \tag{5}$$

where $\tilde{x} := (x_1^{re}, x_1^{im}, \dots, x_m^{re}, x_m^{im})$ and $[x_1, \dots, x_m]_{\otimes h} = R(\tilde{x}) + \sqrt{-1} I(\tilde{x})$.

Theorem 7. *Let $\mathcal{H}$ be a Hermitian tensor, then the following holds.*

(i) If $\mathcal{H}$ is separable, then the kth order relaxation of (5) will give a measure that solves (5) for some k big enough.

(ii) If $\mathcal{H}$ is not separable, then the kth order relaxation of (5) must be infeasible for some k big enough.