Hermitian Tensor Decompositions

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Hermitian Tensors

Definition

A tensor $\mathcal{H} \in \mathbb{C}^{n_1 \times \cdots \times n_m \times n_1 \times \cdots \times n_m}$ is **Hermitian** if
$\mathcal{H}_{i_1 \ldots i_m j_1 \ldots j_m} = \overline{\mathcal{H}_{j_1 \ldots j_m i_1 \ldots i_m}}$ for all $i_1, \ldots, i_m$ and $j_1, \ldots, j_m$.

- A rank-1 Hermitian tensor is in the form

$$\pm[v^1, v^2, \cdots, v^m]_{\otimes h} := \pm\, v^1 \otimes v^2 \cdots \otimes v^m \otimes \overline{v^1} \otimes \overline{v^2} \cdots \otimes \overline{v^m}.$$

- Every Hermitian tensor is a sum of rank-1 Hermitian tensors. The smallest length is the Hermitian rank of $\mathcal{H}$, denoted by $\operatorname{hrank}(\mathcal{H})$.
- Mixed quantum states can be represented by Hermitian tensors.

Basis Hermitian Tensors

For $I = (i_1, \ldots, i_m), J = (j_1, \ldots, j_m)$, $\mathcal{E}^{IJ}(c)$ is the basis Hermitian tensor such that $[\mathcal{E}^{IJ}(c)]_{IJ} = \overline{[\mathcal{E}^{IJ}(c)]_{JI}} = c$ with other entries being 0.

Theorem (Nie-Y.)

When $I \neq J$, then $\operatorname{hrank} \mathcal{E}^{IJ}(c) = 2d$ where d is the number of nonzero entries of $I - J$.

Example

For $I = (1,2)$, $J = (3,4)$ and $c \neq 0$, the basis tensor $\mathcal{E}^{(12)(34)}(c) \in \mathbb{C}^{[4,4]}$ has the Hermitian rank 4, with the following decomposition ($i := \sqrt{-1}$)

$$\frac{1}{4}\left[\begin{pmatrix} c \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}\right]_{\otimes h} + \frac{1}{4}\left[\begin{pmatrix} c \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}\right]_{\otimes h} - \frac{1}{4}\left[\begin{pmatrix} c \\ 0 \\ i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ i \end{pmatrix}\right]_{\otimes h} - \frac{1}{4}\left[\begin{pmatrix} c \\ 0 \\ -i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -i \end{pmatrix}\right]_{\otimes h}.$$

Real Hermitian Tensors

- Real Hermitian tensors are Hermitian tensors whose entries are real.
- **Not** every real Hermitian tensor is $\mathbb{R}$-Hermitian decomposable, i.e. it can be written as a sum of real rank-1 Hermitian tensors.

Theorem (Nie-Y.)

A real Hermitian tensor $\mathcal{H}$ is $\mathbb{R}$-Hermitian decomposable if and only if

$$\mathcal{H}_{i_1 \ldots i_m j_1 \ldots j_m} = \mathcal{H}_{k_1 \ldots k_m l_1 \ldots l_m} \tag{0.1}$$

for all labels such that $\{i_s, j_s\} = \{k_s, l_s\}$ for all $s = 1, \ldots, m$.

Separable Hermitian Tensors

Definition: The Hermitian tensor $\mathcal{H}$ is called *separable* if $\mathcal{H}$ has the positive decomposition $\mathcal{H} = \sum_{i=1}^{r} [u_i^1, \ldots, u_i^m]_{\otimes h}$.

Theorem (Nie-Y.)

$\mathcal{H}$ is separable if and only if there is a Borel measure μ such that

$$\mathcal{H} = \int [x_1, \ldots, x_m]_{\otimes h} \mathrm{d}\mu.$$

- Checking the existence of μ is a truncated moment problem which can be solved by semidefinite relaxations.
- The separability of Hermitian tensors is equivalent to the separability of mixed quantum states.

Thank You Very Much!