Some Properties of $U_q(sl(n))$

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Abstract

This note is devoted to a detailed computation of the commutators of the Hopf algebra $U_q(sl(n))$. It can be considered as a second way to computation the brackets of the Hopf algebra $U_q(sl(n))$ which could be introducing and understanding the $U_q(sl(n))$ for the students

Introduction

Quantum groups, introduced in 1986 by Drinfeld [1], form a certain class of Hopf algebras. U_q to date there is no rigorous, universally accepted definition, but it is generally agreed that this term includes certain deformations in one or more parameters of classical objects associated to algebraic groups, such as enveloping algebras of semisimple Lie algebras or algebras of regular functions on the corresponding algebraic groups. As one can relate algebraic groups with commutative Hopf algebras via group schemes, it is also agreed that the category of quantum groups should correspond to the opposite category of the category of Hopf algebras. This is why some authors define quantum groups as non-commutative and non-cocommutative Hopf algebras.The name quantum group is actually something of a misnomer, since they are not really groups at all. To some extent, quantum groups almost sound like science fiction, especially given the weirdness surrounding the discoveries of quantum physics. So, just what are these exciting new structures called quantum groups? It's always good to be honest at the outset of a significant undertaking. With that said, the reader might be disappointed to learn that there is no rigorous, universally accepted definition of the term quantum group. However, this has not prevented the development of a rich, powerful and elegant theory with an ever broadening horizon of application. Interestingly, there is also a significant collection of examples for which mathematicians in general can say, , that's a quantum group." One of the " of reality is that it appears to be " in the language of mathematics. It is not a rare occasion that a bit of mathematics is developed with no physical application in mind.For more information see [2-6].

The Quantized Enveloping Algebra $U_q(gl(n))$

We fix an invertible element $q \in \mathbb{C}$, $q \neq \pm 1$. So the fraction $\frac{1}{q-q^{-1}}$ is well defined. For any integer n define $[n] := \frac{q^n - q^{-n}}{q-q^{-1}} = q^{n-1} + q^{n-3} + \ldots + q^{-n+3} + q^{-n+1}$ If q is not a root of unity, then $[n] \neq 0$ for any non-zero integer n. If q is a root of unity, then denote its order by d, i.e. $d \in \mathbb{N}$ is minimal such that $q^d = 1$.

We define $U_q(gl(n))$ as a unital associative complex algebra generated by $e_i, f_i, i = 1, 2, \cdots, n-1, k_j, k_j^{-1}, j = 1, 2, \cdots, n$ subject to the relations

$k_i K_j = k_j K_i$; $k_i k_i^{-1} = k_i^{-1} k_i = 1$;

$k_i e_j k_i^{-1} = q^{\frac{\delta_{ij}}{2}} q^{-\frac{\delta_{ij+1}}{2}} e_j$

$k_i f_j k_i^{-1} = q^{-\frac{\delta_{ij}}{2}} q^{\frac{\delta_{ij+1}}{2}} f_j$

$[e_i, f_j] = \delta_{ij} \frac{k_i^2 k_{i+1}^2 - k_{i+1}^2 k_i^2}{q + q^{-1}}$

$[e_i, f_j] = [f_j, e_i] = 0 \ , \mid i - j \mid \geq 2$

$e_i^2 e_{i\pm1} - (q + q^{-1}) e_i e_{i\pm1} e_i + e_{i\pm1} e_i^2 = 0$

$f_i^2 f_{i\pm1} - (q + q^{-1}) f_i f_{i\pm1} f_i + f_{i\pm1} f_i^2 = 0.$

The generators e_i, f_i correspond to the simple roots.

The Lie algebra $sl(n)$

Complex Lie algebras g in general are vector spaces over $\mathbb{C}$ equipped with a nonassociative product, commonly denoted as the Lie bracket. This is a linear map $[,] : g \otimes g \longrightarrow g$ which satisfies

(i) $[a, b] = [b, a]$ antisymmetry,

(ii) $[a, [b, c]] = [[a, b], c] + [b, [a, c]]$ Jacobi identity.

The complex simple Lie algebra $sl(2)$ is spanned as a vector space by three elements X^+,X^- and H. The Lie bracket is given by $[X^+, X^-] = H, [H, X^\pm] = \pm 2X^\pm$.Together with the antisymmetry property and the bilinearity these three relations define the Lie bracket on the whole algebra uniquely.

The special linear Lie algebra of order n (denoted $sl_n(\mathbb{F})$ or $sl(n, \mathbb{F})$) is the Lie algebra of $n \times n$ matrices with trace zero and with the Lie bracket $[X, Y] := XY - YX$ The complex simple Lie algebra $sl(2)$ is spanned as a vector space by three elements X^+,X^- and H. The Lie bracket is given by $[X^+, X^-] = H, [H, X^\pm] = \pm 2X^\pm$.T The special linear Lie algebra of order n (denoted $sl_n(\mathbb{F})$ or $sl(n, \mathbb{F})$) is the Lie algebra of $n \times n$ matrices with trace zero and with the Lie bracket $[X, Y] := XY - YX$.

The Results

Theorem

Let x, y and z be two elements of $U_q(sl(n))$ with q,a and b arbitrary parameters then

(i) $[x, y]_q - [y, x]_q = (1 + q)(xy - yx)$.

(ii) $[x, y]_q + [y, x]_q = (1 - q)(xy + yx)$.

(iii) $[x, y]_q - [y, x]_{q^{-1}} = (1 + q^{-1})xy - (1 + q)yx$.

(iv) $[x, y]_q + [y, x]_{q^{-1}} = (1 - q^{-1})xy + (1 - q)yx$.

(v) $[x, y]_q = [y, x]_q$, *if* $[x, y] = 0$

(vi) $[[y, z]_a, x]_b = [[y, x]_b, z]_a$, *with* $[z, x] = 0$

(vii) $[z, [y, x]_a]_b = [y, [z, x]_b]_a$, *with* $[y, z] = 0$

Theorem

The elements e_{12} and f_{12} of $U_q(sl(3))$ have the bracket

$[e_{12}, f_{12}] = (1 - q^{-1})([e_1, e_2]_q [f_2, f_1] + [f_2, f_1][e_1, e_2]_q) - ([[[e_1, e_2]_q, f_1]_{q^{-1}}, f_2] + [[f_2, [e_1, e_2]_q]_{q^{-1}}, f_1])$

Theorem

The elements h_j and e_{ij} of $U_q(sl(n))$ have the bracket

$[h_j, e_{ij}] = (q - 1)(-a_{ij}(e_i e_{i+1,j} + e_{i+1,j} e_i)) + (e_i [e_{i+1,j}, h_j] + [e_{i+1,j}, h_j] e_i)$.

Theorem

The elements e_{ij} and f_{ij} of $U_q(sl(n))$ have the bracket

$[e_{ij}, f_{ij}] =$ *(1-q)*$([f_i, f_{i+1,j}]_q [e_i, e_{i+1,j}] + [e_i, e_{i+1,j}][f_i, f_{i+1,j}]_q) - ([[[f_i, f_{i+1,j}]_q, e_i]_q, e_{i+1,j}] + [[e_{i+1,j}, [f_i, f_{i+1,j}]_q]_q, e_i])$ *where* $i, j = 1, 2 \cdots n$.

Corollary

For any $n, m \geq 1$ and $k \geq 0$ we have the following

$f_i^n f_{i-1}^k f_i^m \in span\{f_{i-1}^k f_i^{n+m}, f_{i-1}^{k-1} f_i^{n+m} f_{i-1}, \cdots, f_i^{n+m} f_{i-1}^k\}$ *(1)*

and

$f_{i-1}^{n+m} f_i^n \in span\{f_{i-1}^n f_i^n f_{i-1}^m, f_{i-1}^{n-1} f_i^n f_{i-1}^{m+1}, \ldots, f_i^n f_{i-1}^{m+n}\}$ *(2)*

Then $f_{i-1}^{n+1} f_i^n = f_{i-1}^{n+1-l} f_i^n f_{i-1}^l$ *and* $f_i^{n+1} f_{i-1}^n = f_i^{n+1-l} f_{i-1}^n f_i^l$.

Conclusion

In this short paper, we have supple new technique of f the commutators of the Hopf algebra $U_q(sl(n))$ with an approach suitable for university students. With elementary mathematical tools, in our opinion, it is possible to understand that this new manner of computing.

References

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