Some Properties of $U_q(sl(n))$

Mehsin Jabel Atteya
Department of Mathematics, College of Education,
Al-Mustansiriyah University, Iraq.
E-mail: mehsinatteya88@uomustansiriyah.edu.iq

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ABSTRACT
This note is devoted to a detailed computation of the commutators of the Hopf algebra $U_q(sl(n))$. It can be considered as a second way to computation the brackets of the Hopf algebra $U_q(sl(n))$ which could be introducing and understanding the $U_q(sl(n))$ for the students.

1. Introduction

Quantum groups, introduced in 1986 by Drinfeld [1], form a certain class of Hopf algebras. U_q to date there is no rigorous, universally accepted definition, but it is generally agreed that this term includes certain deformations in one or more parameters of classical objects associated to algebraic groups, such as enveloping algebras of semisimple Lie algebras or algebras of regular functions on the corresponding algebraic groups. For more information see [2] and [3].

2. The Quantized Enveloping Algebra $U_q(gl(n))$

The special linear Lie algebra of order n (denoted $sl_n(\mathbb{F})$ or $sl(n, \mathbb{F})$) is the Lie algebra of $n \times n$ matrices with trace zero and with the Lie bracket $[X, Y] := XY - YX$
We fix an invertible element $q \in \mathbb{C}$, $q \neq 1$. So the fraction $\frac{1}{q-q^{-1}}$ is well defined. For any integer n define $[n] := \frac{q^n - q^{-n}}{q-q^{-1}} = q^{n-1} + q^{n-3} + ... + q^{-n+3} + q^{-n+1}$.
If q is not a root of unity, then $[n] \neq 0$ for any non-zero integer n. If q is a root of unity, then denote its order by d, i.e. $d \in \mathbb{N}$ is minimal such that $q^d = 1$.
We define $U_q(gl(n))$ as a unital associative complex algebra generated by $e_i, f_i, i = 1, 2, \cdots, n-1, k_j, k_j^{-1}, j = 1, 2, \cdots, n$ subject to the relations

$$k_i K_j = k_j K_i;\ k_i k_i^{-1} = k_i^{-1} k_i = 1;\ k_i e_j k_i^{-1} = q^{\delta \frac{ij}{2}} q^{-\delta \frac{ij+1}{2}} e_j,\ k_i f_j k_i^{-1} = q^{-\delta \frac{ij}{2}} q^{\delta \frac{ij+1}{2}} f_j$$

$$[e_i, f_j] = \delta_{ij} \frac{k_i^2 k_{i+1}^2 - k_{i+1}^2 k_i^2}{q + q^{-1}},\ [e_i, f_j] = [f_j, e_i] = 0\ ,\ \mid i - j \mid \geq 2$$

$e_i^2 e_{i\pm1} - (q + q^{-1}) e_i e_{i\pm1} e_i + e_{i\pm1} e_i^2 = 0$, $f_i^2 f_{i\pm1} - (q + q^{-1}) f_i f_{i\pm1} f_i + f_{i\pm1} f_i^2 = 0$. The generators e_i, f_i correspond to the simple roots.

3. The Main Results

Theorem 3.1. *Let x, y and z be elements of $U_q(sl(n))$ with q,a and b arbitrary parameters then*

(i) $[x,y]_q - [y,x]_q = (1+q)(xy - yx)$.

(ii) $[x,y]_q + [y,x]_q = (1-q)(xy + yx)$.

(iii) $[x,y]_q - [y,x]_{q^{-1}} = (1+q^{-1})xy - (1+q)yx$.

(iv) $[x,y]_q + [y,x]_{q^{-1}} = (1-q^{-1})xy + (1-q)yx$.

(v) $[x,y]_q$=$[y,x]_q$, *if* $[x,y] = 0$

(vi) $[[y,z]_a, x]_b = [[y,x]_b, z]_a$, *with* $[z,x] = 0$

(vii) $[z,[y,x]_a]_b = [y,[z,x]_b]_a$, *with* $[y,z] = 0$

Proposition 3.2. *The elements e_{12} and f_{12} of $U_q(sl(3))$ have the bracket*

$[e_{12}, f_{12}] = (1 - q^{-1})([e_1,e_2]_q[f_2,f_1] + [f_2,f_1][e_1,e_2]_q) - ([[[e_1,e_2]_q, f_1]_{q^{-1}}, f_2] + [[f_2,[e_1,e_2]_q]_{q^{-1}}, f_1])$

Theorem 3.3. *The elements e_{ij} and f_{ij} of $U_q(sl(n))$ have the bracket*

$[e_{ij}, f_{ij}] = (1-q)([f_i, f_{i+1,j}]_q[e_i, e_{i+1,j}] + [e_i, e_{i+1,j}][f_i, f_{i+1,j}]_q) - ([[[f_i, f_{i+1,j}]_q, e_i]_q, e_{i+1,j}] + [[e_{i+1,j}, [f_i, f_{i+1,j}]_q]_q, e_i])$ *where* $i,j = 1,2\cdots n$.

Theorem 3.4. *The elements h_j and e_{ij} of $U_q(sl(n))$ have the bracket*
$[h_j, e_{ij}] = (q-1)(-a_{ij}(e_ie_{i+1,j} + e_{i+1,j}e_i)) + (e_i[e_{i+1,j}, h_j] + [e_{i+1,j}, h_j]e_i)$.

Depend on the same technique of the proof of above theorem ,we can achieve the following

Theorem 3.5. *The elements h_j and f_{ij} of $U_q(sl(n))$ have the bracket*
$[h_j, f_{ij}] = (q-1)(-a_{ij}(f_if_{i+1,j} + f_{i+1,j}f_i)) + (f_i[f_{i+1,j}, h_j] + [f_{i+1,j}, h_j]f_i)$.

References

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