Quantum Majorization in Infinite Dimensions

Priyanga Ganesan (priyanga.g@tamu.edu)

Ph.D student, Department of Mathematics, Texas A&M University

Classical majorization

Majorization is a mathematical tool that is used to compare disorderedness in statistics, physics, economics and computer science.

Definition *Let $u, v \in \mathbb{R}^n$. We say u majorizes v (denoted $v \prec u$) if there exists a doubly stochastic matrix $A \in M_n(\mathbb{R})$ such that $v = Au$.*

Importance in Quantum Information Theory

In Quantum Information Theory (QIT), majorization is used in the study of entanglement transformations to determine when a given state can be transformed into another via local operations and classical communications (LOCC).

Theorem (Nielsen, 1999) *Let H_A, H_B be finite dimensional Hilbert spaces. For two bipartite pure states $\phi, \psi \in B(H_A \otimes H_B)$,*

$$|\phi\rangle\langle\phi| \xrightarrow{LOCC} |\psi\rangle\langle\psi| \iff \lambda_\Psi \prec \lambda_\Phi$$

where λ_Ψ and λ_Φ are the vectors containing Schmidt numbers of Ψ and Φ respectively.

Quantum majorization in finite dimensions

Matrix majorization has been generalised to quantum majorization between bipartite states to accommodate the ordering of states and processes in quantum mechanical systems.

Definition (Quantum Majorization) *Let H_A, H_B be finite dimensional Hilbert spaces. Let $\rho, \sigma \in B(H_A \otimes H_B)$ be two compatible bipartite quantum states (i.e. $\rho^A = \sigma^A$). We say ρ quantum majorizes σ (denoted $\sigma \prec_q \rho$) if there exists a quantum channel (completely positive trace preserving linear map) $\mathcal{E} : B(H_B) \to B(H_B)$ such that $(I_A \otimes \mathcal{E})\rho = \sigma$.*

Entropic characterization of quantum majorization

Recently, an entropic characterization of quantum majorization was given using conditional min-entropy with applications to thermodynamics.

The conditional min entropy of a bipartite state $\Omega \in B(H_A \otimes H_B)$ is defined as

$$H_{min}(A|B)_\Omega := -log \inf_{\tau_B \geq 0} \{Tr[\tau_B] : I_A \otimes \tau_B \geq \Omega\}$$

Theorem (Gour-Jennings-Buscemi-Duan-Marvian, 2018)

$$\sigma \prec_q \rho \iff H_{min}(A|B)_{(\Phi \otimes I_B)\rho} \leq H_{min}(A'|B)_{(\Phi \otimes I_B)\sigma}$$

for any system A' with $dim(A) = dim(A')$ and all quantum channels $\Phi : B(H_A) \to B(H_{A'})$.

Objective

- **Goal:** We extend the definition and characterization of quantum majorization to infinite dimensional Hilbert spaces.
- **Motivation:** Thermodynamical systems have an infinite number of energy levels and real-world physical thermal processes could be described using infinite dimensions. Hence, it is of interest to study quantum majorization in the infinite-dimensional setting.

Extension to infinite dimensions

Let H_A, H_B be two infinite dimensional Hilbert spaces. Let $\rho, \sigma \in \mathcal{T}(H_A \otimes H_B)$ be two compatible bipartite quantum states, where $\mathcal{T}(H)$ denotes bounded trace class operators on a Hilbert space H.

Definition *ρ quantum majorizes σ if there exists a quantum channel $\mathcal{E} : \mathcal{T}(H_B) \to \mathcal{T}(H_B)$ such that $(I_A \otimes \mathcal{E})\rho = \sigma$.*

Entropic characterization in infinite dimensions

Theorem *For two bipartite density operators $\rho, \sigma \in \mathcal{T}(H_A \otimes H_B)$, ρ quantum majorizes σ if and only if for any entanglement breaking channel $\Phi : \mathcal{T}(H_A) \to \mathcal{T}(H_A)$,*

$$H_{min}(A|B)_{\Phi \otimes I_B(\rho)} \leq H_{min}(A|B)_{\Phi \otimes I_B(\sigma)}$$

Outline of proof:

$\Rightarrow$ One direction follows from data processing inequality satisfied by conditional min-entropy.

$$H_{min}(A|B)_\rho \leq H_{min}(A|B)_{I_A \otimes \mathcal{E}(\rho)}$$

$\Leftarrow$ The converse implication uses a Hahn-Banach separation argument and properties of operator space projective tensor product.

- Operator space duality: $(E \hat{\otimes} F)^* \cong CB(E, F^*)$ for operator spaces E & F.

$$B(H_A \otimes H_B) = (\mathcal{T}(H_A \otimes H_B))^* = (\mathcal{T}(H_A) \hat{\otimes} \mathcal{T}(H_B))^* = CB(\mathcal{T}(H_A), B(H_B))$$

- Connection between conditional min-entropy and operator space projective tensor norm:

$$H_{min}(A|B)_\rho = -\log \|\rho\|_{\mathcal{T}(H_A) \hat{\otimes} B(H_B)}$$

Applications

1. **Quantum Interpolation:** For two family of density operators $\{\rho\}_{i \in \mathbb{N}}$ and $\{\sigma\}_{i \in \mathbb{N}}$ in $B(H_B)$, there exists a quantum channel such that $\Phi(\rho_i) = \sigma_i$ for all $i \in \mathbb{N}$ if and only if for any finitely supported probability distribution λ_i on $\mathbb{N}$ and any set of density operators $\{\omega_i\}_{i \in \mathbb{N}} \in B(H_A)$,
$$H_{min}(A|B)_{\sum_i \lambda_i \omega_i \otimes \rho_i} \leq H_{min}(A|B)_{\sum_i \lambda_i \omega_i \otimes \sigma_i}$$
2. **Channel Factorization:** For two quantum channels $T, S : \mathcal{T}(H_B) \to \mathcal{T}(H_B)$, there exists a quantum channel Φ such that $\Phi \circ T = S$ if and only if for any separable density operator $\rho \in \mathcal{T}(H_A \otimes H_B)$,
$$H_{min}(A|B)_{I_A \otimes T(\rho)} \leq H_{min}(A|B)_{I_A \otimes S(\rho)}$$