Free Probability Exercises

1. Suppose that A and B are $*$-free noncommunative probability spaces inside a noncommutative probability space $(\mathcal{A}, \varphi)$. Let $a_1, \cdots, a_n \in A$ and $b_1, \cdots, b_n \in B$ and $n \geq 2$. Assume that $a_2, \cdots, a_n$ and $b_1, \cdots, b_{n-1}$ are all centered. Show that $a_1 b_1 a_2 b_2 \cdots a_n b_n$ is centered (note: we are **not** assuming that a_1 and b_n are centered)

2. Let $(\mathcal{A}, \varphi)$ be a C*-probability space. Suppose $u \in \mathcal{A}$ is a centered unitary and $a \in \mathcal{A}$ is $*$-free from u.

   (a) Show that the law of a is the same as the law of u^*au.

   (b) Show that a is $*$-free from u^*au.

3. In this exercise, we explore the sum of two "free fair coins," i.e. the law of $v + w$ where v and w are freely independent self-adjoint, centered unitaries. Before proceeding, ask yourself why the term "fair coin" applies to the individual laws of v and w. In what follows, $(\mathcal{A}, \varphi)$ is a C*-probability space.

   (a) Let u be a Haar unitary in $\mathcal{A}$. Show that the distribution of $u + u^*$ is the **arcsine law**:

   $$\mathbb{1}_{[-2,2]} \frac{1}{\pi\sqrt{4 - t^2}} dt.$$

   (Hint: To start, argue why $\varphi[(u + u^*)^n] = \frac{1}{2\pi} \int_0^{2\pi} (2\cos(t))^n dt = \frac{1}{\pi} \int_0^{\pi} (2\cos(t))^n dt$. From there, use a substitution.)

   (b) Let v and w be self-adjoint centered unitary elements in $(\mathcal{A}, \varphi)$ which are $*$-free from each other. Show that vw is a Haar unitary in $\mathcal{A}$.

   (c) Show that wvw is a centered self-adjoint unitary which is $*$-free from v (this quickly follows from which exercise?).

   (d) Show that $vw + wv$ and $wvw + v$ have the same moments and hence the same law (since both elements are self-adjoint). Deduce that the law of a sum of two free centered self-adjoint unitary elements (for instance $w + v$) is also the arcsine law. (Hint: First show that $(vw + wv)^{2k} = (wvw + v)^{2k}$ for each positive integer k. Then, argue that every odd moment of $wvw + v$ is 0).

   The above exercise shows a theme in free probability: combining free elements often "kills off" atoms from the law. In this exercise, two free elements with atomic distributions were added together to produce a distribution with no atoms anywhere. Can this happen in the classically independent case?

4. The overlying point of the following exercise is to show that free independence and commutativity only coincide under the most trivial of circumstances, namely when at least one element is a scalar multiple of the identity.

   (a) Show that if a and b are $*$-free in a $*$-probability space $(\mathcal{A}, \varphi)$, then

   $$\varphi(a^*b^*ab) = |\varphi(a)|^2\varphi(b^*b) + \varphi(a^*a)|\varphi(b)|^2 - |\varphi(a)|^2 \cdot |\varphi(b)|^2$$

   (try to use exercise 1 to make the expansion of $\varphi(a^*b^*ab)$ a bit easier to work with.)

   (b) Let a and b be as in part (a) above, and suppose that in addition to being $*$-free, a and b^* commute. Use part (a) to show that

   $$\varphi(a^*a)\varphi(b^*b) = |\varphi(a)|^2\varphi(b^*b) + \varphi(a^*a)|\varphi(b)|^2 - |\varphi(a)|^2 \cdot |\varphi(b)|^2$$

   and use this equation to show that this means

   $$\varphi([a - \varphi(a)1]^*[a - \varphi(a)1])\varphi([b - \varphi(b)1]^*[b - \varphi(b)1]) = 0.$$

   Deduce that if φ is faithful and a and b^* are free elements that commute, then at least one of a or b is a scalar multiple of the identity.

5. Let $x_1, \cdots, x_n$ be a free family of self-adjoint semicircular elements with variance 1 in a C*-probability space $(\mathcal{A}, \varphi)$. Show that

   $$\frac{x_1 + \cdots + x_n}{\sqrt{n}}$$

   is semicircular with variance 1.

   Hint: Please don't do this by computing all of the moments. You will hate yourself! Instead, ask yourself if there is a good spatial representation of $x_1, \cdots, x_n$.

   Note: This exercise is meant to demonstrate the stability of the semicircular law under addition of freely independent elements. This is in analogy to the stability of the Gaussian law under sums of classically independent elements.

6. Let $(\mathcal{A}, \varphi)$ and $(\mathcal{B}, \psi)$ be (C* or W*) probability spaces generated by $a = (a_1, \cdots, a_n)$ and $b = (b_1, \cdots, b_n)$ respectively. Suppose φ and ψ are faithful and that the laws μ_a and μ_b are equal. Show that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic as (C* or W*) algebras.

   Hint: Think spatially. There should be a natural unitary between the GNS Hilbert spaces. What does this unitary do in regards to the $\mathcal{A}$ and $\mathcal{B}$ GNS representations? As a comment, this exercise shows that the joint law of a tuple of elements uniquely determines the C* or W* algebra said tuple generates

7. In this exercise, we explore a matricial model for freeness due to Shlyakhtenko to show a somewhat surprising result (See, for example, Theorem 4.4 in Dykema's "Free Group Factors" article in the Free Probability and Operator Algebras edition of Munster Lectures in Mathematics).

   $$(L(\mathbb{Z}), \tau) * (M_2(\mathbb{C}), \mathrm{tr}) \cong (L(\mathbb{F}_4), \tau) \otimes M_2(\mathbb{C}).$$

Let $\mathcal{H}$ be a Hilbert space, and $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ be an orthonormal set in $\mathcal{H}$. Let $(\mathcal{A}, \psi) = (\mathcal{B}(\mathcal{F}(\mathcal{H})), \varphi) \otimes (M_2(\mathbb{C}), \mathrm{tr})$ where $\varphi(x) = \langle x\Omega, \Omega\rangle$. Define $L \in \mathcal{A}$ by

$$L = \frac{1}{\sqrt{2}} \begin{pmatrix} \ell(\xi_1) & \ell(\xi_2) \\ \ell(\xi_3) & \ell(\xi_4) \end{pmatrix}$$

(a) Show that $L^*L = 1$ but that $LL^* \neq 1$ in $\mathcal{A}$.

(b) Suppose that n and m are nonnegative integers that are not both zero, show that $\psi(L^n(L^*)^m)) = 0$. Deduce that the law of $L + L^*$ with respect to ψ is semicircular of variance 1.

(c) Let X be any trace zero 2×2 matrix with scalar entries. Show that $L^*XL = 0$.

(d) Use the results above to show that $W^*(L)$ and $M_2(\mathbb{C})$ are $*$-free (Hint, centered elements in $W^*(L)$ are weak limits of linear combinations of what words?). In particular, this exercise implies that $W^*(L + L^*)$ is free from $M_2(\mathbb{C})$

(e) Show that $e_{11}(W^*(\{L + L^*\} \cup M_2(\mathbb{C}))e_{11}$ is generated as a von Neumann algebra by:

$$\left\{ \begin{pmatrix} \ell(\xi_1) + \ell(\xi_1)^* & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \ell(\xi_2) + \ell(\xi_3)^* & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \ell(\xi_4) + \ell(\xi_4)^* & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

(f) Show that the real and imaginary parts of $\ell(\xi_2) + \ell(\xi_3)^*$ are freely independent semicircular elements that are also free from $\ell(\xi_1)$ and $\ell(\xi_4)$ under φ. (Note: $\ell(\xi_2) + \ell(\xi_3)^*$ is an example of a **circular element**: one whose real and imganiary parts are free identically distributed semicircular elements)

(g) Deduce that

$$(L(\mathbb{Z}), \tau) * (M_2(\mathbb{C}), \mathrm{tr}) \cong (L(\mathbb{F}_4), \tau) \otimes M_2(\mathbb{C})$$

This exercise illustrates another free probabilistic theme: compressing free products increases the number of free generators.