Problem session exercises

1. Show that the functional τ on $\mathcal{M} \rtimes G$ given by $\tau(x) = \langle x(\widehat{1} \otimes \delta_e), \widehat{1} \otimes \delta_e \rangle$ is a normal faithful tracial state (the harder part is faithfulness, for this you should define an "algebra of right multiplication" by $\mathcal{M} \rtimes_{\text{alg}} G$ in the commutant of $\mathcal{M} \rtimes G$ for which $\widehat{1} \otimes \delta_e$ is cyclic).

2. Prove that in a tracial crossed product $\mathcal{M} \rtimes G$, every $x \in \mathcal{M} \rtimes G$ admits a unique Fourier expansion $x = \sum g \in G x_g u_g$ for some $x_g \in \mathcal{M}$ so the sum converges in $\|\cdot\|_2$-norm. Moreover, show $x_g = E_{\mathcal{M}}(x u_g^*)$ (where $E_{\mathcal{M}} : \mathcal{M} \rtimes G \to \mathcal{M}$ is the trace preserving conditional expectation). (Hint: Consider $\widehat{x} \in L^2(\mathcal{M}) \otimes \ell^2(G)$).

3. We say an action $G \curvearrowright^{\sigma} \mathcal{M}$ is properly outer (or free) if for each $g \in G$, 0 is the only $y \in \mathcal{M}$ such that $y\sigma_g(x) = xy$ for every $x \in \mathcal{M}$. Prove that a trace preserving action $G \curvearrowright^{\sigma} (\mathcal{M}, \tau)$ is properly outer if and only if $\mathcal{Z}(\mathcal{M}) = \mathcal{M}' \cap \mathcal{M} \rtimes G$. (Hint: use the Fourier expansion for $y \in \mathcal{M}' \cap \mathcal{M} \rtimes G$ and compare the Fourier coefficients of $xy = yx$).

4. Let $G \curvearrowright (\mathcal{M}, \tau)$ be a trace preserving action. Notice that the center $\mathcal{Z}(\mathcal{M})$ is G-invariant. Assume that the action is free (as in Problem 3.), then show that $\mathcal{M} \rtimes G$ is a factor if and only if the induced action $G \curvearrowright \mathcal{Z}(\mathcal{M})$ is ergodic (i.e. $\{x \in \mathcal{M} : \sigma_g(x) = x\,,\ \forall g \in G\} = \mathbb{C}1$).

5. Constructing the Gaussian von Neumann algebra via GNS. Let $\mathcal{H}_{\mathbb{R}}$ be a real Hilbert space and define a $*$-algebra $\mathcal{D}_0 = \text{span}\{w(\xi) : \xi \in \mathcal{H}_{\mathbb{R}}\}$ with product and adjoint given by $w(\xi)w(\eta) = w(\xi + \eta)$ and $w(\xi)^* = w(-\xi)$ for all $\xi, \eta \in \mathcal{H}_{\mathbb{R}}$.
   (a) Show $\tau : \mathcal{D}_0 \to \mathbb{C}$ given by extending $\tau(w(\xi)) = e^{-\|\xi\|^2}$ is a unital, positive and faithful linear functional. (Hint: you will need to use that for any $\xi_1, ..., \xi_k \in \mathcal{H}_{\mathbb{R}}$ the matrix $\left[e^{-\|\xi_i - \xi_j\|^2}\right]_{i,j}$ is strictly positive definite).
   (b) Take the vector space $\widehat{\mathcal{D}_0}$ with the $\|\cdot\|_2$-norm from τ. Show that for each $\xi \in \mathcal{H}_{\mathbb{R}}$ the map $\widehat{w(\eta)} \mapsto w(\xi)\widehat{w(\eta)} = \widehat{w(\xi + \eta)}$ extends to a unitary $w(\xi) \in \mathbb{B}(L^2(\mathcal{D}_0, \tau))$.
   (c) Sketch the proof that τ extends to a normal faithful tracial state on $\mathcal{D} = \mathcal{D}_0'' \subseteq \mathbb{B}(L^2(\mathcal{D}_0, \tau))$.

6. (Bonus) In this exercise you will study the unitary representation of G induced by the Gaussian construction (which will determine the $L(G)$-bimodule structure of $\widetilde{\mathcal{M}} = \mathcal{D} \rtimes G$)
   Let $\mathcal{H}_{\mathbb{R}}$ be a Real Hilbert space and $\pi : G \to \mathcal{O}(\mathcal{H}_{\mathbb{R}})$ an orthogonal representation
   (a) Take the complexified Hilbert space $\mathcal{H}_{\mathbb{C}} = \mathcal{H}_{\mathbb{R}} \otimes \mathbb{C}$ (where $\langle \mu\xi, \nu\eta \rangle = \mu\bar{\eta}\langle \xi, \eta \rangle$ for all $\mu, \nu \in \mathbb{C}$ and $\xi, \eta \in \mathcal{H}_{\mathbb{R}}$, identifying $\mu\xi := \xi \otimes \mu$). Verify this is a Hilbert space.
   (b) Denote the one dimensional Hilbert space by $\mathcal{H}_{\mathbb{C}}^{\odot 0} = \mathbb{C}\Omega$ (with $\|\Omega\| = 1$) and for $n \in \mathbb{N}_{>0}$ let $\mathcal{H}_{\mathbb{C}}^{\odot n}$ be the symmetrized n-fold tensor power of $\mathcal{H}_{\mathbb{C}}$. Prove that for each $\xi \in \mathcal{H}_{\mathbb{R}}$ we can define $\text{Exp}(\xi) = \sum_{n=0}^{\infty} (n!)^{-1/2} \xi^{\otimes n} \in \oplus_{\mathbb{N}} \mathcal{H}_{\mathbb{C}}^{\odot n}$ (where $\xi^{\otimes 0} = \Omega$) and this vector exponentiation satisfies $\langle \text{Exp}(\xi), \text{Exp}(\eta) \rangle = e^{\langle \xi, \eta \rangle}$.
   (c) We call $\mathcal{SH}_{\mathbb{C}} := \oplus_{\mathbb{N}} \mathcal{H}_{\mathbb{C}}^{\odot n}$ the symmetrized (or bosonic) Fock space. Show that the set $\{\text{Exp}(\xi) : \xi \in \mathcal{H}_{\mathbb{R}}\}$ is linearly independent and total in $\mathcal{SH}_{\mathbb{C}}$. (This is more of a remark, the proof can be found in [Gui72, Proposition 2.2]).

(d) For $\xi \in \mathcal{H}_{\mathbb{R}}$ define the map $w(\xi)$ by $\mathrm{Exp}(\eta) \mapsto e^{-\|\xi\|^2 - \sqrt{2}\langle \xi, \eta \rangle} \mathrm{Exp}(\eta + \sqrt{2}\xi)$. Show that the operators $\{w(\xi) : \xi \in \mathcal{H}_{\mathbb{R}}\}$ extend to unitaries on $\mathcal{S}\mathcal{H}_{\mathbb{C}}$ such that $w : \mathcal{H}_{\mathbb{R}} \to \mathscr{U}(\mathcal{S}\mathcal{H}_{\mathbb{C}})$ is a group homomorphism.

(e) Show that Ω is a cyclic and separating vector for $\mathcal{D} = \mathrm{span}\{w(\xi) : \xi \in \mathcal{H}_{\mathbb{R}}\}$.

(f) Show that the orthogonal representation π induces a unitary representation of G on $\mathcal{S}\mathcal{H}_{\mathbb{C}}$.