GOALS, July 23, 2024

Exercises on groupoids and their *-algebras

1. Use the groupoid axioms given in lecture to check at least 1-2 of the following facts for γ, α, β in a groupoid G:
   (a) $(\gamma^{-1})^{-1} = \gamma$
   (b) If $\alpha = \gamma^{-1}\gamma$, then $\alpha \in G^{(0)}$.
   (c) If $\alpha^2 = \alpha$, then $\alpha \in G^{(0)}$
   (d) $\alpha\gamma = \beta\gamma$ implies $\alpha = \beta$

2. The *transformation groupoid* for a set X and group Γ which acts on X via bijections is defined in the following way:

$$G := \Gamma \times X$$
$$G^{(0)} := \{e\} \times X \text{ (which is identified with } X \text{ below)}$$
$$r(g, x) := g \cdot x; \quad s(g, x) := x$$
$$(g, h \cdot x)(h, x) := (gh, x) \text{ (and these are the composable pairs)}$$

   Find $(g, x)^{-1}$. Also check some of the groupoid axioms, such as $\gamma\gamma^{-1} = r(\gamma)$, $\gamma^{-1}\gamma = s(\gamma)$, or that multiplication is associative.

3. Show that if G is étale, for each $x \in G^{(0)}$, xG and Gx are discrete, i.e. singletons are open in the relative topology.

4. Check that if $f \in C_c(G)$ is supported on a bijection, then $f^* * f$ is supported on $s(\mathrm{supp} f)$. Calculate $(f^* * f)(\gamma)$ for $\gamma \in \mathrm{supp} f$. Also calculate $f * g$ for $g \in C_c(G^{(0)})$.

5. Check that for each $\eta \in G$, the operator defined by

$$U_\eta : \ell^2(Gs(\eta)) \to \ell^2(Gr(\eta))$$
$$\delta_\gamma \mapsto \delta_{\gamma\eta^{-1}}$$

   is a unitary operator, and that the relation

$$\pi_{r(\eta)} = U_\eta \pi_{s(\eta)} U_\eta^*$$

   on the regular representations π_x for $x \in G^{(0)}$ is satisfied.