NOTES ON C*-ALGEBRAS

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Contents

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1. A first look at C*-algebras

Preview of Lecture: To help guide your reading, we indicate here which of the following material we will address in lecture and which we will assume familiarity with:

Much of the material in this chapter will be brushed over fairly quickly, with the goal of getting to our first marquee theorem, the Gelfand–Naimark Theorem (Theorem 2.1). You may want to read this chapter on your own, before or after lecture, to solidify your understanding of how these fundamental ideas are used in the proof of Theorem 2.1.

Proposition 1.29 is a fundamental, and surprising, result about C*-algebras, which you should definitely read for yourself if we don't have time to discuss it in lecture.

In a Banach space, there is often additional algebraic structure, in particular multiplication.

Definition 1.1. A *Banach *-algebra A* is a multiplicative involutive Banach space whose norm satisfies the following:

$$\|ab\| \leq \|a\|\|b\|$$

for all $a, b \in A$. A multiplicative, linear, *-preserving map between Banach *-algebras is called a **-homomorphism*. A bijective *-homomorphism is an **-isomorphism*.

Ideally, we'd like involution on a Banach algebra to also be isometric. This and other magical results follow from the additional assumption that the norm $\|\cdot\|$ on A satisfies the C**-identity*:

$$\|a^*a\| = \|a\|^2$$

for all $a \in A$. It follows from this that

$$\|a\|^2 = \|a^*a\| \leq \|a^*\|\|a\|,$$

and hence that $\|a\| \leq \|a^*\| \leq \|a^{**}\| = \|a\|$.

Definition 1.2. A C**-algebra* is a Banach $*$-algebra whose norm satisfies the C*-identity.

Remark. *Calling these* C**-algebras is already highly suggestive. In fact, when they were first introduced, they were called B^*-algebras, and the notion of* C**-algebra was reserved for norm closed $*$-subalgebras of $B(\mathcal{H})$. In the coming days, we shall justify calling these* C**-algebras, but for the sake of not encouraging archaic terminology, we take the privilege before we earn it.*

Recall from the Prerequisite Notes exercises that the norm on $B(\mathcal{H})$ satisfies the C*-identity, meaning any closed self-adjoint subspace of $B(\mathcal{H})$ is a C*-algebra. These are known as *concrete* C**-algebras*.

Example 1.3. Recall the unilateral shift $S \in B(\ell^2(\mathbb{N}))$ from the Prerequisite Notes. The norm closure of the $*$-algebra generated by S in $B(\ell^2(\mathbb{N}))$ is a C*-algebra often called the *Toeplitz algebra*.

Exercise 1.4. Let X be a locally compact Hausdorff topological space. We denote by $C_0(X)$ the space of all continuous $\mathbb{C}$-valued functions on X vanishing at infinity[1]. Show this is a C*-algebra with involution given by complex conjugation ($f^*(x) = \overline{f(x)}\ \forall x \in X$) and norm given by the sup norm ($\|f\| = \sup_{x\in X} |f(x)|$).

Example 1.5. Consider the C*-algebra $C(\mathbb{T})$ consisting of all continuous functions on the compact Hausdorff space $\mathbb{T} = \{\lambda \in \mathbb{C} \ : \ |\lambda| = 1\}$ (sometimes denoted S^1). (Why don't we say $C_0(\mathbb{T})$?) It follows from the Stone-Weierstraß approximation theorem ([6, I.5,6]) that Laurent polynomials, i.e. polynomials of the form $\sum_{k=-n}^{n} \alpha_n z^n$, are dense in $C(\mathbb{T})$. Moreover, these form a dense *-subalgebra, and $C(\mathbb{T})$ is actually the C*-algebra generated by the function $f \in C(\mathbb{T})$ given by $f(z) = z$.

As is often the case, a C*-algebra A is a little more friendly to work with when it has an identity element, $1 \in A$. In this case, we call A *unital*. Because A has an involution and a norm, if A is unital, then its unit must satisfy:

(1) $1^* = 1^*1 = 11^* = 1$, and

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[1]Recall that this means such a function f is small off of compact subsets of X, i.e., for any $\varepsilon > 0$ there exists a $K \subset X$ compact so that $\sup_{K^c} |f(x)| < \varepsilon$. For example, we can think of $C_0(\mathbb{R})$ as the continuous functions on $\mathbb{R}$ whose limit towards $\pm\infty$ is zero.

(2) $\|1\| = 1$.

Analogous to elements in $B(\mathcal{H})$, we call an element a in a C*-algebra A

- *normal* if $a^*a = aa^*$,
- *self-adjoint* if $a = a^*$,
- a *projection* if $a = a^* = a^2$,
- a *unitary* if $a^*a = aa^* = 1$,
- an *isometry* if $a^*a = 1$,
- a *partial isometry* if $a = aa^*a$.

Exercise 1.6. Describe the projections in $C_0(X)$ where X is

(1) $(0, 1]$,
(2) $[0, 1]$,
(3) $[0, 1/3] \cup [2/3, 1]$,
(4) $[0, 1/3) \cup (2/3, 1]$.

Note (Check) that for any element a in a C*-algebra is the sum of two self-adjoint operators, its real and imaginary parts:

$$\mathrm{Re}(a) = \frac{1}{2}(a + a^*) \qquad \mathrm{Im}(a) = \frac{1}{2i}(a - a^*). \tag{1.1}$$

This useful decomposition lets us reduce many questions to the case of self-adjoint operators.

Proposition 1.7. *A linear map between* C*-*algebras is* $*$*-preserving iff it maps self-adjoint elements to self-adjoint elements.*

Proof. A *-preserving map clearly sends self-adjoint elements to self-adjoint elements. We want to show the reverse implication. Let $\phi : A \to B$ be a linear map that map self-adjoint elements to self-adjoint elements. For $a \in A$, we write $a = \mathrm{Re}(a) + i\mathrm{Im}(a)$ and $a^* = \mathrm{Re}(a) - i\mathrm{Im}(a)$. By linearity,

$$\phi(a) = \phi(\mathrm{Re}(a)) + i\phi(\mathrm{Im}(a))$$
$$\phi(a^*) = \phi(\mathrm{Re}(a)) - i\phi(\mathrm{Im}(a)).$$

Since $\mathrm{Re}(a)$ and $\mathrm{Im}(a)$ are self-adjoint, $\phi(\mathrm{Re}(a))$ and $\phi(\mathrm{Im}(a))$ are self-adjoint by assumption. So

$$\phi(a)^* = \phi(\mathrm{Re}(a) + i\mathrm{Im}(a))^* = (\phi(\mathrm{Re}(a)) + i\phi(\mathrm{Im}(a)))^* = \phi(\mathrm{Re}(a)) - i\phi(\mathrm{Im}(a)) = \phi(a^*). \square$$

1.1. **Unitizations and Spectra.** Let us briefly recap and expand on some facts about the spectrum of an operator in a Banach algebra, with an eye towards C*-algebras.

An element a in a unital algebra is *invertible* when there exists another element b in the algebra that acts as a left and right inverse, i.e. $ab = ba = 1$. We write $GL(A)$ for the set of invertible elements of A.

Sometimes, when an element has a left inverse, it is automatically a right inverse. In particular, this is the case for matrix algebras. In fact, a matrix $T \in \mathrm{M}_n(\mathbb{C})$ is invertible if and only if it is injective, i.e. if and only if $\ker(T) = \{0\}$. In infinite dimensions, this is certainly still a necessary condition, but it is no longer sufficient alone.

Exercise 1.8. Give an example of an operator on $B(\ell^2(\mathbb{N}))$ that is injective but not invertible.

Fortunately, the Open Mapping Theorem gives us some guidance on what needs to be satisfied:

Corollary 1.9 (to OMT/Inverse Function Theorem)**.** *For a Hilbert space* $\mathcal{H}$*,* $T \in B(\mathcal{H})$ *is invertible iff* T *is bijective.*

Example 1.10. Unitary operators are important classes of invertible operators. In fact, the set of unitaries in a unital C*-algebra A forms a subgroup $\mathcal{U}(A)$ of the group of invertible elements, $GL(A)$.

With the notion of invertibility, we can define the spectrum of a given element a in a unital C*-algebra A.

$$\sigma(a) := \{\lambda \in \mathbb{C} \ : \ \lambda 1 - a \notin GL(A)\}.$$

Observe that if a is invertible then $0 \notin \sigma(a)$.

Remark 1.11. Unlike when $A = \mathrm{M}_n(\mathbb{C})$, the elements of $\sigma(a)$ are not all eigenvalues. That is, we can have $\lambda \in \sigma(a)$ without having $\ker(\lambda 1 - a) \neq \{0\}$. (Can you give an example?)

Example 1.12. If A is a unital C*-algebra and $u \in A$ is a unitary, then $\sigma(u) \subset \mathbb{T}$.

Proof. First note that for any invertible operator $a \in A$, the spectrum of the inverse is the inverse of the spectrum. To see this, fix an invertible a, and note that $a \in GL(A)$ means $0 \notin \sigma(a)$ and $0 \notin \sigma(a^{-1})$. For $\lambda \neq 0$, if $\lambda - a$ is invertible, then so is $\lambda^{-1}a^{-1}(\lambda - a) = a^{-1} - \lambda^{-1}$ and vice versa, i.e., for any $\lambda \in \mathbb{C}$, $\lambda \in \sigma(a) \iff \lambda^{-1} \in \sigma(a^{-1})$.

Now for any $\lambda \in \sigma(u)$, we have that $\lambda^{-1} \in \sigma(u^{-1}) = \sigma(u^*)$. Since u^* is also a unitary, we know $\|u\|^2 = \|u^*u\| = 1 = \|uu^*\| = \|u^*\|^2$, and so Theorem 3.17 from the Prereqs tells us that $|\lambda| \leq 1$ and $|\lambda^{-1}| \leq 1$. We conclude that $|\lambda| = 1$. □

Exercise 1.13. Recall from Example 3.13 in the Prerequisite Material that a continuous function f on a locally compact and Hausdorff space X is invertible if $1/f$ is continuous on X. For the C*-algebra $C(\mathbb{T})$, what type of operator is the generator $f(z) = z$? What is the spectrum of $f(z) = z$ in $C(\mathbb{T})$?

But not all C*-algebras have units. One important example is $\mathcal{K}(\mathcal{H})$ when $\mathcal{H}$ is infinite dimensional. (What about when it's finite dimensional?)

Proposition 1.14. *If $\mathcal{H}$ is infinite dimensional, then $K(\mathcal{H})$ is nonunital.*

The naïve proof here would be to say that, as a proper ideal, $K(\mathcal{H})$ cannot contain the unit of $B(\mathcal{H})$. However, this does not rule out $K(\mathcal{H})$ having unit of its own.

Proof. Suppose $K(\mathcal{H})$ has a unit $I \in K(\mathcal{H})$, i.e. $Ia\xi = aI\xi = a\xi$ for all $a \in K(\mathcal{H})$ and $\xi \in \mathcal{H}$. Let $\eta \in \mathcal{H}$, and choose any unit vector $\xi \in \mathcal{H}$. Then

$$I\eta = I\langle \xi, \xi\rangle \eta = I\Theta_{\eta,\xi}\xi = \Theta_{\eta,\xi}\xi = \langle \xi, \xi\rangle \eta = \eta.$$

Hence $I\eta = \eta$ for all $\eta \in \mathcal{H}$. But then $I = 1_{B(\mathcal{H})}$, which is not in the proper ideal $K(\mathcal{H})$, and we have reached a contradiction. □

Another important class of nonunital examples comes from spaces of continuous functions.

Exercise 1.15. For a locally compact topological Hausdorff space X, when is the C*-algebra $C_0(X)$ unital? What is the unit? Can you think of interesting classes of non-unital algebras?

So, how can we make sense of a spectrum in the nonunital setting? We just add a unit! Well, technically, we embed A into a unital C*-algebra.

Definition 1.16. The "smallest" unital C*-algebra containing A is called its *unitization*, $\tilde{A}$. For a non-unital C*-algebra A, we define $\tilde{A}$ as follows:

$$\tilde{A} := A \oplus \mathbb{C}$$

with algebraic operations given by

$$\begin{aligned}(a,\alpha)(b,\beta) &= (ab + \alpha b + \beta a, \alpha\beta)\\ (a,\alpha)^* &= (a^*, \bar{\lambda})\\ \|(a,\alpha)\| &= \sup_{b\in A, \|b\|\leq 1} \|ab + \alpha b\|\end{aligned}$$

Why does this norm make $\tilde{A}$ a Banach algebra, much less a C^*-algebra? See Proposition 1.18 below.

This definition does not feel intuitive the first time around, but it does guarantee that $1_{\tilde{A}} := (0, 1)$ is actually a unit for $\tilde{A}$. Odd definitions are often best understood by seeing what motivated them (i.e., what phenomena they are isolating and abstracting). Consider the following examples.

Exercise 1.17.

(1) If $A \subset B(\mathcal{H})$ is a C*-subalgebra of $B(\mathcal{H})$ that does not contain a unit, you can "unitize" it by just taking the C*-algebra generated by A and $1_{\mathcal{H}}$:

$$\mathrm{C}^*(A, 1_{\mathcal{H}}) = \{a + \lambda 1_{\mathcal{H}} \ : \ \lambda \in \mathbb{C}, a \in A\}.$$

What would multiplication/scalar addition look like here?

Convince yourself that there is an isometric $*$-isomorphism between $\mathrm{C}^*(A, 1_{\mathcal{H}})$ and $\tilde{A}$.

For the norm, the argument is faster after a little more theory. (Revisit after Remark 1.26.)

(2) Identify

$$C_0((0,1]) \leftrightarrow \{f \in C([0,1]) \ : \ f(0) = 0\} \subset C([0,1]).$$

By taking the closure of the subalgebra of $C([0,1])$ generated by $C_0((0,1])$ and the constant function 1, we get $C([0,1])$. (Why do we need to take the closure?)

That is, $C([0,1])$ should be the unitization of $C_0((0,1])$. Show that, indeed, $C([0,1]) \cong \widetilde{C_0((0,1])}$.

Because of the example from $B(\mathcal{H})$, even in an abstract setting, elements of $\tilde{A}$ are often written as $a + \lambda 1_{\tilde{A}}$ as opposed to (a, λ).

Proposition 1.18. *Any nonunital* C*-*algebra A embeds into the unital* C*-*algebra $\tilde{A}$ as an ideal of codimension* 1*, i.e. no other proper ideal of $\tilde{A}$ contains A and $\tilde{A}/A = \mathbb{C}$.*

Remark 1.19. Just to reiterate, unless otherwise specified, all of our ideals are 2-sided.

Proof. That $\tilde{A}$ is a unital $*$-algebra is readily verified. To see that the norm is a Banach algebra norm, notice that it is exactly the norm induced from $B(A)$ where we identify $a \in A$ with the left multiplication operator $L_a \in B(A)$ given by $L_a(b) = ab$, and we identify (a, α) with $L_a + \alpha \mathrm{id}_A$. In other words, the norm on $\tilde{A}$ is the norm induced from $B(A)$ on the $*$-subalgebra of operators $\{L_a + \alpha \mathrm{id}_A : a \in A, \alpha \in \mathbb{C}\}$. Moreover, note that the identification $a \mapsto L_a$ is isometric. Indeed, using the C*-identity, we have for any nonzero $a \in A$,

$$\|a\| = \|a\left(\frac{a^*}{\|a\|}\right)\| \le \sup_{\|b\| \le 1} \|ab\| \le \|a\| \sup_{\|b\|\le 1} \|b\| \le \|a\|.$$

So, $\|(a, 0)\| = \|a\|$, and the embedding of A into $\tilde{A}$ is isometric. Since $B(A)$ is complete (by Exercise 2.4 from the Prerequisite notes), $\{L_a + \alpha \mathrm{id}_A : a \in A, \alpha \in \mathbb{C}\}$ is complete, and so $\tilde{A}$ is a Banach algebra. By design, A is an ideal of of $\tilde{A}$ of codimension 1.

It remains to show that the given norm satisfies the C*-identity. To that end, we compute for $a \in A$ and $\alpha \in \mathbb{C}$

$$\begin{aligned}
\|(a,\alpha)\|^2 &= \sup_{\|b\|\le 1} \|ab + \alpha b\|^2 \\
&= \sup_{\|b\|\le 1} \|b^*(a^*ab + \alpha a^* b + \bar{\alpha} ab + |\alpha|^2 b)\| \\
&\le \sup_{\|b\|\le 1} \|a^*ab + \alpha a^* b + \bar{\alpha} ab + |\alpha|^2 b\| \\
&= \|(a,\alpha)^*(a,\alpha)\| \le \|(a,\alpha)^*\| \|(a,\alpha)\|.
\end{aligned}$$

So $\|(a,\alpha)\| \le \|(a,\alpha)^*\|$, and a symmetric argument yields $\|(a,\alpha)^*\| = \|(a,\alpha)\|$. Then the above inequality gives

$$\|(a,\alpha)\|^2 \le \|(a,\alpha)^*(a,\alpha)\| \le \|(a,\alpha)\|^2. \qquad \square$$

Remark 1.20. Secretly, this is not so much of an embedding of A into $B(A)$, but rather an embedding of A into the C*-subalgebra of $B(A)$ consisting of "multipliers," which we call the *multiplier algebra of* A, which is denoted by $M(A)$. This is the largest (nondegenerate) unitization of A in the sense that any unital C*-algebra containing A as a "nondegenerate" (the term is *essential*) ideal embeds canonically into $M(A)$. The multiplier algebra is universal in this sense but nonetheless highly constructible. It can appear as a subalgebra of $B(A)$, as the idealizer of A in A^{**} or in any nondegenerate faithful representation of A, or as the adjointable operators on the Hilbert module $A \otimes \ell^2$. It is an extremely useful big object to stick your C*-algebra in when you need more "room". See https://pskoufra.info.yorku.ca/files/2016/07/Multiplier-Algebras.pdf for a nice introduction.

Remark 1.21. What if A was already unital? Sometimes it is notationally convenient to adopt the convention that $A = \tilde{A}$ when A is unital (staying consistent with the idea that $\tilde{A}$ is the "smallest" unital C*-algebra containing A).

However, sometimes we want to add a unit on top of the natural unit for the C*-algebra in a construction called "forced unitization" where A is still embedded as a maximal ideal in $A \oplus \mathbb{C}$, and the unit of A becomes just the projection $1_A \oplus 0$. When dealing with a forced unitization, it is better to take a slightly different perspective. Suppose A is unital with unit 1_A, and consider $A \oplus \mathbb{C}$ with the usual norm, involution, and pointwise multiplication, i.e.,

$$(a, \alpha)(b, \beta) = (ab, \alpha\beta)$$
$$(a, \alpha)^* = (a^*, \bar{\alpha})$$
$$\|(a, \alpha)\| = \max\{\|a\|, \|\alpha\|\}.$$

This is a unital C*-algebra (what's the unit?). Moreover, $A \oplus \mathbb{C}$ contains A as a two-sided norm-closed ideal with co-dimension 1.

Less abstractly, consider a unital C*-algebra B and a C*-subalgebra $A \subset B$ which is unital, i.e., there exists $1_A \in A$ that acts as a unit on A, but we need not have $1_A = 1_B$. However, the unit 1_A of A must be a projection in B, call it p, which has an orthogonal complement $q := 1_B - p \in B$. The forced unitization of A in this setting is not so much adding 1_B to A but taking $A \oplus \mathbb{C}q$ inside B. (Note that $qa = 0 = aq$ for all $a \in A$, so this direct sum does make sense in B.)

One thing that makes unitizations nice to work with is that a $*$-homomorphism always has a unique and natural extension to the unitization.

Proposition 1.22. *Let A, B be C*-algebras with A non-unital and $\pi : A \to B$ a $*$-homomorphism. Then there is a unique extension of π to a unital $*$-homomorphism $\tilde{\pi} : \tilde{A} \to \tilde{B}$ given by $\tilde{\pi}(a + \lambda 1_{\tilde{A}}) = \pi(a) + \lambda 1_{\tilde{B}}$.*

Proof. We just need to check that the given formula is a $*$-homomorphism. Linearity and $*$-preserving are immediate. For $a, b \in A$ and $\lambda, \eta \in \mathbb{C}$, we compute

$$\begin{aligned}\tilde{\pi}(a + \lambda 1_{\tilde{A}})\tilde{\pi}(b + \eta 1_{\tilde{A}}) &= (\pi(a) + \lambda 1_{\tilde{B}})(\pi(b) + \eta 1_B) \\ &= \pi(ab) + \lambda\pi(b) + \eta\pi(a) + \lambda\eta 1_{\tilde{B}} = \tilde{\pi}(ab + \lambda b + \eta a + \lambda\eta 1_{\tilde{A}}).\end{aligned}$$

The uniqueness is forced by the fact that we require $\tilde{\pi}$ to be linear and $1_{\tilde{A}} \mapsto 1_{\tilde{B}}$. Indeed, if $\psi : \tilde{A} \to \tilde{B}$ is another unital extension of π, then for each $a + \lambda 1_{\tilde{A}} \in \tilde{A}$, we have

$$\psi(a + \lambda 1_{\tilde{A}}) = \psi(a) + \psi(\lambda 1_{\tilde{A}}) = \pi(a) + \lambda 1_{\tilde{B}} = \tilde{\pi}(a + \lambda 1_{\tilde{A}}). \qquad \square$$

Note that we didn't actually use the fact that π was $*$-preserving in the proof. That's a good thing, because in the next chapter, we will need the following extension of Proposition 1.22.

Exercise 1.23. Let A, B be C*-algebras and $\pi : A \to B$ a multiplicative and linear map. Prove that there is a unique multiplicative linear map $\tilde{\pi} : \tilde{A} \to \tilde{B}$ satisfying $\tilde{\pi}(a + \lambda 1_{\tilde{A}}) = \pi(a) + \lambda 1_{\tilde{B}}$.

Remark 1.24. If A is unital and $\pi : A \to B$ is a $*$-homomorphism, then $\pi(1_A)$ is a projection in B and $\pi(A)$ is a unital *-subalgebra. (In fact, it is a C*-algebra, but we need to establish a little more before we can say this.)

With the ability to canonically and minimally unitize, we can still define the spectrum of an element in any C*-algebra.

Definition 1.25. The *spectrum* of an element in a C*-algebra is

$$\sigma(a) := \{\lambda \in \mathbb{C} \ : \ \lambda 1_{\tilde{A}} - a \notin GL(\tilde{A})\}$$

where $A = \tilde{A}$ when A is unital.

Remark 1.26. Suppose A is a non-unital C*-subalgebra of a unital C*-algebra B. Then there is a clear $*$-preserving bijective homomorphism between $\tilde{A}$ and $C^*(A, 1_B)$ given by $(a, \alpha) \mapsto a + \alpha$. With Proposition 1.29 this means that, when a unit is available in an ambient C*-algebra, unitizing A is just adjoining that unit. Of course, there is now the problem that for any $a \in A$, its spectrum in A might be larger than its spectrum in B (an element has more potential inverses in B). We will see later that this is not the case.

Now that we have a notion of spectra for unital and nonunital C*-algebras, we are ready to see two consequences of the C*-identity that are, quite frankly, magic.

First we recall Theorems 3.17 and 3.21 from the Prerequisite material:

Theorem. *For any element a in Banach algebra A, $\sigma(a)$ is a nonempty compact subset of $\mathbb{C}$. Moreover, the spectrum of a is contained in the closed ball $\{\lambda \in \mathbb{C} : |\lambda| \leq \|a\|\}$. In particular, this means that $r(a) \leq \|a\|$ where $r(a) = \sup_{\lambda\in\sigma(a)} |\lambda|$ is the spectral radius of a.*

Remark 1.27. This implies the very useful fact that for any element a in a unital Banach algebra with $\|a\| < 1$, the element $1 - a$ is invertible with inverse $\sum_{n\geq 0} a^n$, or alternatively if b satisfies $\|1 - b\| < 1$, then b is invertible.

Theorem. *For any element a in Banach algebra A,*

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

When our Banach algebra A is a C*-algebra, it turns out the norm of any normal element *is* its spectral radius.

Lemma 1.28. *For any normal element a in a C*-algebra A,*

$$\|a\| = r(a).$$

Proof. First, we assume that $a = a^*$. Then repeated use of the C*-identity for a, i.e. $\|a\|^2 = \|a^2\|$, tells us that

$$r(a) = \lim_n \|a^{2^n}\|^{2^{-n}} = \|a\|.$$

Now, suppose a is normal. Then

$$\begin{aligned}\|a\|^2 &= \|a^*a\| = r(a^*a)\\ &= \lim_n \|(a^*a)^n\|^{1/n} = \lim_n \|(a^n)^*a^n\|^{1/n} = \lim_n \|a^n\|^{2/n}\\ &= r(a)^2.\end{aligned}$$

□

As a Banach $*$-algebras, we consider C*-algebras "the same" when they are $*$-isomorphic, i.e. there exists a $*$-preserving bijective homomorphism between them. Normally, for a Banach space, you'd also request that the bijective linear map be isometric. For $*$-isomorphisms between C*-algebras, this will be automatic, thanks again to the C*-identity.

Proposition 1.29. *A $*$-homomorphism $\pi : A \to B$ between unital C*-algebras is contractive (i.e. $\|\pi\| \leq 1$) and hence continuous. A $*$-isomorphism between C*-algebras is isometric.*

Proof. Suppose $\pi : A \to B$ is a $*$-homomorphism. Let $a \in A$. Then a^*a is a normal element in A, which means $\|a^*a\| = r(a^*a)$. Since homomorphisms (or their unitizations) preserve invertibility, $r(\pi(a^*a)) \leq r(a^*a)$. This is where the C*-norm comes in:

$$\|a\|^2 = \|a^*a\| = r(a^*a) \geq r(\pi(a^*a)) = r(\pi(a)^*\pi(a)) = \|\pi(a)^*\pi(a)\| = \|\pi(a)\|^2.$$

If π is a $*$-isomorphism, then a symmetric argument shows the inequality above is an equality. □

Exercise 1.30. Show that Proposition 1.29 also holds in the case where A and B are non-unital. What about when A is non-unital but B is unital?

Remark 1.31. The case where A is unital but B is non-unital (or likewise when A and B are unital and $\pi(1_A) \neq \pi(1_B)$) is a little more delicate. (Though, note that this cannot happen when π is surjective since $\pi(1_A)$ will be the unit for $\pi(A) = B$.) The following argument will let us conclude nonetheless that $r(\pi(a^*a)) \leq r(a^*a)$ for any $a \in A$.

Suppose B_0 is a *-subalgebra of a unital C*-algebra B (e.g., $\pi(A)$), and B_0 contains a unit p that is not the unit of B, i.e., p is a projection in B that acts as a unit on B_0. For $a \in B_0$, let $\sigma_{B_0}(a)$ denote the set of $\lambda \in \mathbb{C}$ such that there exists no $b \in B_0$ with $(a - \lambda p)b = p = b(a - \lambda p)$, i.e., the spectrum of a inside B_0, and let $\sigma_B(a)$ denote the spectrum of a inside B, i.e., the set of $\lambda \in \mathbb{C}$ such that there exists no $c \in B$ with $(a - \lambda 1_B)c = 1_B = c(a - \lambda 1_B)$. Let $q = 1_B - p$. If $\lambda \neq 0$ and $b \in B_0$ is such that $b(a - \lambda p) = p = (a - \lambda p)b$, then $(b - \lambda^{-1}q)$ is the inverse of $(a - \lambda 1_B)$. Likewise, if $a - \lambda 1_B$ is invertible with inverse $b \in B$, then pbp gives you the inverse in B_0. It follows that $\sigma_{B_0}(a)\backslash\{0\} = \sigma_B(a)\backslash\{0\}$.

It follows from Lemma 1.28 that in C*-algebras the algebraic structure determines the norm:

$$\|x\| = (\|x^*x\|)^{1/2} = (r(x^*x))^{1/2}$$

(Compare with the same fact for matrices.) It follows from this that a C*-algebra carries a unique norm making it a C*-algebra.

Remark 1.32. What this is saying is that if $(A, \|\cdot\|)$ is a C*-algebra and $\|\cdot\|'$ is another C*-norm on A (without assuming A is complete with respect to $\|\cdot\|'$), then $\|x\| = \|x\|'$ for all $x \in A$.

There's a subtlety here that can sometimes be a little tricky. If A were just a $*$-algebra, i.e., not completed with respect to some norm, then we can often define multiple distinct C*-norms on A so that the completion of A with respect to these norms becomes a C*-algebra. In fact, the question of when various *-algebras admit only one unique C*-norm is tied in with some of the deepest problems in C*-theory. This idea will come up again in Sections 12 and 5.

We will be able to say more about $*$-homomorphisms once we have established more on C*-ideals.

2. Commutative C*-algebras

Preview of Lecture: To help guide your reading, we indicate here which of the following material we will address in lecture and which we will assume familiarity with:

The main goal in this lecture is proving the Gelfand-Naimark theorem for commutative C*-algebras (Theorem 2.1) and introducing the Functional Calculus (Corollary 2.30).

To that end, we will use without proof all of the results in Section 1. We will introduce the unitization from Section 1, but with more focus on the intuition in Remark 1.26.

From Section 2, we use without proof the correspondence between maximal ideals and characters established in Definition 2.6 - Corollary 2.16.

We will prove Lemma 2.21 and assume its corollary, Lemma 2.23, to complete the proof of Theorem 2.1.

Proposition 2.27 and Corollary 2.28 establish the important fact that the spectrum of an element in a C*-algebra is independent of the ambient unital C*-algebra. However, we will bypass this argument in lecture and go straight for a description of the correspondence in the Functional Calculus (Corollary 2.30).

2.1. **The Gelfand-Naimark Theorem: A discussion.** Some of you may have heard of the study of C*-algebras described as "non-commutative topology" or "non-commutative continuous functions". This perspective is really what jump-started the interest in C*-algebras in the first place, and it comes from the following theorem, which is the focal point of this section:

Theorem 2.1 (Gelfand-Naimark Theorem)**.** *Any commutative* C*-*algebra* A *is isometrically* $*$*-isomorphic to the* C*-*algebra* $C_0(X)$ *for some locally compact Hausdorff space* X*. Moreover, when* A *is unital,* X *is compact.*

Remark 2.2. We saw in Exercise 1.4 that for any locally compact Hausdorff space X, the algebra $C_0(X)$ with the sup norm is a C*-algebra. So, this theorem actually gives the following 1-1 correspondence:

$$\left\{\begin{array}{c}\text{Commutative}\\ \text{C}^*\text{-algebras}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}C_0(X),\\ X \text{ locally compact}\\ \text{Hausdorff}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{locally compact}\\ \text{Hausdorff}\\ \text{spaces}\end{array}\right\}$$

Not only that, but we can also say that any isometric *-isomorphism between commutative C*-algebras corresponds to a homeomorphism between their associated locally compact Hausdorff spaces, i.e., $C_0(X) \cong C_0(Y)$ iff $X \cong Y$.

We also get a nice correspondence in the unital setting:

$$\left\{\begin{array}{c}\text{Commutative}\\ \text{unital}\\ \text{C}^*\text{-algebras}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}C(X),\\ X \text{ compact}\\ \text{Hausdorff}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{compact}\\ \text{Hausdorff}\\ \text{spaces}\end{array}\right\}$$

Exercise 2.3. Let X and Y be homeomorphic compact Hausdorff spaces. Show that $C(X) \cong C(Y)$.

Recall from Example 2.2 from the Prerequisite notes that $C_0(X)$ is all the functions on X that *vanish at infinity.* Since our space X will be locally compact and Hausdorff, we encourage the intuition from Remark 2.3, where we consider "infinity" to be the additional point $\{\infty\}$ in the one-point compactification of X. In this setting, we think of $C_0(X)$ as the subspace

$$\{f \in C(X \cup \{\infty\}) \mid f(\infty) = 0\}$$

of its unitization $C(X \cup \{\infty\})$.

So what is this locally compact Hausdorff space X, and how do we view our commutative C*-algebra A as continuous functions on X? This sort of phenomenon is actually quite common in mathematics, and it arises in the presence of dual spaces (as in Section 2.1 from the Prerequisite Notes).

Consider, for now, a Banach space $\mathcal{X}$. Its dual $\mathcal{X}^*$ is also a Banach space, which in turn has its own dual $(\mathcal{X}^*)^* := \mathcal{X}^{**}$ called the *double dual* of $\mathcal{X}$. The double dual of $\mathcal{X}$ is of particular interest because $\mathcal{X}$ embeds naturally into $\mathcal{X}^{**}$, i.e., we can actually just view $\mathcal{X}$ as linear functionals on $\mathcal{X}^*$. Indeed, any $x \in \mathcal{X}$ induces a linear functional ev_x on $\mathcal{X}^*$ given by "point-evaluation:"

$$ev_x(\phi) := \phi(x), \forall\ \phi \in \mathcal{X}^*.$$

Exercise 2.4. Check that ev_x does indeed define a linear functional on $\mathcal{X}^*$. Show moreover that ev_x is continuous with respect to the weak*-topology on $\mathcal{X}^*$ (Definition 2.18 in Prerequisites).

It turns out that the identification $x \leftrightarrow ev_x$ gives an isometric embedding of $\mathcal{X}$ into $\mathcal{X}^{**}$. (For a proof, we refer you to [19]. Theorem 4.3 gives the argument and the prologue to the start of section 4.5 tells you how to interpret Theorem 4.3.)

Theorem 2.5. *For any Banach space $\mathcal{X}$, there is a canonical isometric embedding $\iota : \mathcal{X} \hookrightarrow \mathcal{X}^{**}$ given by $\iota(x) = ev_x$. Moreover, $\iota(\mathcal{X})$ consists of exactly the linear functionals on $\mathcal{X}^*$ that are also continuous in the weak*-topology on $\mathcal{X}^*$.*

We often drop the extra notation and just identify $\mathcal{X}$ with $\iota(\mathcal{X})$.[2]

That means we can already view a C*-algebra A (which is also a Banach space) as the space of weak*-continuous $\mathbb{C}$-valued linear functionals on A^*. (Why do we care about weak*-continuity? Because all compactness arguments go through the Banach-Alaoglu Theorem (Theorem 2.20 in Prerequisites).) However, this identification is on the level of Banach spaces, whereas we are after an identification on the level of C*-algebras (that means we want an isometric *-isomorphism between C*-algebras, not just an isometric linear map between Banach spaces). Since this requires more structure, we will need to refine our approach. Still, the upshot thus far is that in order to find our locally compact Hausdorff space X, we will look to $A^*_{\leq 1}$ endowed with the weak*-topology.

To get an idea of where to go next, we consider the examples of commutative C*-algebras that we understand quite well: $C_0(X)$ for X locally compact and Hausdorff. Notice (check) that each $t \in X$ induces a linear functional $ev_t : C_0(X) \to \mathbb{C}$ given by $f \mapsto f(t)$ for all $f \in C_0(X)$.

It's a Corollary of the Hahn–Banach Theorem (Corollary 2.25 in the Prerequisites) that for any normed vector space $\mathcal{V}$, $\mathcal{V}^*$ separates the points of $\mathcal{V}$. (**Exercise:** Prove this! What's the seminorm in question? the subspace $\mathcal{W}$?) That is, for any $f, g \in C_0(X)$, $f = g \iff \phi(f) = \phi(g)$ for all $\phi \in C_0(X)^*$.

However, you may recall from Section 2.1 in the Prerequisite notes, the dual space of $C_0(X)$ is a little …involved[3], and in this case, it is overkill. Indeed, for any $f, g \in C_0(X)$, we have $f = g \iff ev_t(f) = ev_t(g)$ for all $t \in X$. So, in this case, to really capture $C_0(X)$, we only needed some very special linear functionals.

Likewise, when trying to capture the information of a general commutative Banach *-algebra, we will also restrict to a very special class of linear functionals, which are called characters, a term that you may have heard in an algebra course with nary a norm in sight.

Definition 2.6. A *character* on a Banach *-algebra A is a multiplicative linear functional on A. The *spectrum* of a commutative Banach *-algebra A, denoted $\hat{A}$, is the set of all nonzero characters on A equipped with the weak*-topology. (Hence $\hat{A}$ is often also called the *character space* for A.)

Notice that for any given $a \in A, \phi \in \hat{A}$ we may have $\phi(a) = 0$; we simply can't have $\phi(a) = 0$ for all $a \in A$ if we want $\phi \in \hat{A}$.

Exercise 2.7. Let X be a locally compact Hausdorff space and $t \in X$. Show that ev_t is a character on $C_0(X)$.

For the following proposition, and the rest of this section, recall that $\hat{A} \subseteq A^* = B(A, \mathbb{C})$ always comes with a norm, namely the operator norm: $\|\phi\| := \sup\{|\phi(a)| : a \in A, \|a\| \leq 1\}$.

Exercise 2.8. How is the operator norm on A^* related to the weak-$*$ topology?

Proposition 2.9. *Let A be a commutative C*-algebra. Then $\hat{A} \cup \{0\}$ is a weak-$*$ compact subset of the unit ball of A^*, where 0 denotes the zero functional that maps $a \mapsto 0$ for all $a \in A$. When A is unital, $\hat{A}$ is weak-$*$ compact.*

In particular, $\hat{A}$ is a locally compact Hausdorff space, which is compact when A is unital.

Proof. First we check that $\hat{A} \subset A^*_{\leq 1}$. Let $\phi \in \hat{A}$. Suppose $\|\phi\| > 1$; then there exists $a \in A$ with $\|a\| < 1$ and $\phi(a) = 1$. Since $\|a\| < 1$, $1 - a$ is invertible in $\tilde{A}$. So, we use the unique unital extension of ϕ to $\tilde{A}$ (Exercise

[2] This sort of identification happens in many areas of mathematics. It is super cool, but takes a little while to get your head around. Also, ask Brent about bra-ket notation.

[3] I own up to my C*-prejudice here.

1.23) and compute

$$1 = \phi((1-a)(1-a)^{-1}) = (\phi(1)-\phi(a))\phi((1-a)^{-1}) = (0)\phi((1-a)^{-1}) = 0,$$

which is an obvious contradiction.

Now, since $\hat{A} \cup \{0\}$ is contained in the unit ball of A^*, by the Banach–Alaoglu Theorem (Theorem 2.20 in the Prerequisite notes), all we need to show is that it is weak-$*$ closed. To that end, suppose we have a net $(\phi_i)_{i\in I}$ of characters (multiplicative linear functionals) that converges weak-$*$ to some bounded linear functional $\phi \in A^*$. We need to check that ϕ is multiplicative, but this follows from the fact that pointwise multiplication is continuous. Indeed, for any $a, b \in A$, we have

$$\phi(ab) = \lim_i \phi_i(ab) = \lim_i \phi_i(a)\phi_i(b) = \lim_i \phi_i(a) \lim_i \phi_i(b) = \phi(a)\phi(b).$$

(**Exercise:** Why do we need to work with $\hat{A} \cup \{0\}$ here? Try to think of an example where we need 0.) It follows that $\hat{A} \cup \{0\}$ is a compact Hausdorff space (with respect to the weak-$*$ topology).

Note that if A is unital, then for any $\phi \in \hat{A}$, we have $\phi(1) = \phi(1^2) = \phi(1)^2$, so $\phi(1) = 1$. This means we cannot have a net $(\phi_i)_{i\in I}$ of characters which converges weak-$*$ to 0. (**Exercise:** Prove this!) Therefore, $\hat{A}$ is itself a weak-$*$ closed subset of the unit ball in A^*. □

So, now we have our (locally) compact Hausdorff space $X = \hat{A}$ for which we want an isometric *-isomorphism $A \cong C_0(X)$. Moreover, we already have a natural map $A \to C_0(\hat{A})$ in mind:

Definition 2.10. For a commutative C*-algebra A, we define the *Gelfand transform* $\Gamma : A \to C_0(\hat{A})$ by $\Gamma(a)(\phi) = \phi(a)$ for all $\phi \in \hat{A}$, i.e. $\Gamma(a) = ev_a \in A^{**}$ is the point evaluation at a.

If you are wearing your skeptical hat, you may notice that we haven't yet proved that $\Gamma(a) \in C_0(\hat{A})$ for all $a \in A$! This will be the first order of business in the next section. So if you're willing to take that on faith, we can give the real statement of the Gelfand–Naimark Theorem.

Theorem 2.11 (Gelfand–Naimark)**.** *For any commutative* C**-algebra A, the Gelfand transform is an isometric $*$-isomorphism of A onto $C_0(\hat{A})$.*

Notice that if A is unital, then $C_0(\hat{A}) = C(\hat{A})$.

Remark 2.12. As a consequence of the Gelfand–Naimark theorem, we will show later (Corollary 2.25) that characters on a commutative C*-algebra are automatically $*$-preserving. We assume for now that they are just homomorphisms. Indeed, much of the theory we develop on our way to the Gelfand–Naimark theorem holds in general for Banach algebras, although we restrict our focus here to C*-algebras.

2.2. **Proof of the Gelfand–Naimark Theorem.** Before we get into the technical part of the proof of the Gelfand-Naimark Theorem, let's at least check that the Gelfand transform is a reasonable candidate for a $*$-isomorphism:

Exercise 2.13. For any (not necessarily commutative) C*-algebra A, the Gelfand transform $\Gamma : A \to C(\hat{A})$ is an algebra homomorphism. Moreover, $\|\Gamma(a)\| \le \|a\|$.

(In proving this, make sure you check that $\Gamma(a)$ is indeed continuous for all $a \in A$!)

It is not so easy to see that the range of Γ is a subset of $C_0(\hat{A})$. Let $a \in A$ and $\phi \in A^*$. Then $\Gamma(a) : A^* \to A^*$ given by $\Gamma(a)(\phi) := \phi(a)$ is a well-defined function, makes sense as a function on all of A^*, not just on $\hat{A}$. Moreover, it turns out that $\hat{A} \cup \{0\}$ is the one-point compactification of $\hat{A}$ in A^*. Once you are convinced of that, and you recall that

$$C_0(\hat{A}) = \{f : \hat{A} \cup \{0\} \to \mathbb{C} \text{ continuous } | f(0) = 0\},$$

then one simply observes that (with our newly extended definition of $\Gamma(a)$ as a function on A^*) for any $a \in A$,

$$\Gamma(a)(0) = 0(a) = 0.$$

So the key difficulty is to show that $\hat{A} \cup \{0\}$ is indeed the one-point compactification of $\hat{A}$.

To prove this, we first need to discuss maximal ideals, i.e., ideals that are not contained in any other proper ideal. It turns out there is a one-to-one correspondence between maximal ideals in A and characters.

Exercise 2.14. A maximal ideal in a unital C*-algebra is automatically closed.
(Hint: If $J \subset A$ is a proper ideal, consider $\overline{J} \cap B(1_A, 1)$.)

Exercise 2.15 (Gelfand–Mazur)**.** If A is a simple, unital, commutative Banach algebra, then $A = \mathbb{C}$.

Corollary 2.16. *If A is a unital commutative Banach algebra, then any maximal ideal in A has co-dimension 1, i.e. if $J \subset A$ is a maximal ideal, then $A/J \cong \mathbb{C}$.*

Proof. If $J \subset A$ is a maximal ideal, then A/J is simple. The rest follows from Gelfand–Mazur. □

Before we get too far into the weeds, here is an exercise to build intuition.

Exercise 2.17.

(1) Show that all maximal ideals in $C([0,1])$ are of the form $\{f \in C([0,1]) \,:\, f(t) = 0\}$ for some $t \in [0,1]$.
(2) For each $t \in [0,1]$, define the map $ev_t : C([0,1]) \to \mathbb{C}$ by $ev_t(f) = f(t)$. Show that $\widehat{C([0,1])} = \{ev_t \,:\, t \in [0,1]\}$.
(3) Recall that for $A = C_0((0,1])$, its unitization is $\tilde{A} := C([0,1])$. That means we can identify $C_0((0,1])$ with a maximal ideal inside $C([0,1])$. To which character does $C_0((0,1])$ correspond?
Show that this character agrees with the functional $\phi_0 : \tilde{A} \to \mathbb{C}$ given by $\phi_0(f + \lambda 1) = \lambda$ for all $f \in A$.

Lemma 2.18. *For any maximal ideal J in a unital* C*-*algebra A, the quotient map $\phi_J : A \to A/J$ is a character. Moreover, the correspondence $J \mapsto \phi_J$ is a bijection.*

Proof. From Corollary 2.16 and Theorem 3.8 in the Prerequisite notes, each maximal ideal $J \triangleleft A$ gives a continuous homomorphism $\phi_J : A \to \mathbb{C} \cong A/J$. Moreover, the quotient map ϕ_J is norm-decreasing. Consequently, $\phi_J \in \hat{A}$ for each maximal ideal J. Furthermore, if J, K are distinct maximal ideals, then ϕ_J, ϕ_K have distinct kernels, so $\phi_J \neq \phi_K$. We conclude that the map $J \mapsto \phi_J$ is injective.

On the other hand, if $\phi \in \hat{A}$, then the multiplicativity and boundedness/continuity of ϕ implies that $\ker \phi$ is a 2-sided norm-closed ideal. As we observed already in the proof of Proposition 2.9, we must have $\phi(1) = 1$, so $\ker \phi \neq A$. Since $\phi : A \to \mathbb{C}$ is therefore onto, $\ker \phi$ is an ideal of co-dimension 1, and so it is automatically a maximal ideal. It follows that the map $J \mapsto \phi_J$ is onto, and hence a bijection. □

How do we get a similar bijection in the non-unital case? Recall that when A is not unital, it embeds into $\tilde{A}$ as an ideal with co-dimension 1. In fact, A is the kernel of the character $\phi_0 : \tilde{A} \to \tilde{A}/A = \mathbb{C}$ given by $\phi_0(a, \lambda) = \lambda$. (**Exercise:** Check the assertions above, including the statement that ϕ_0 is a character.) Notice that when restricted to A, ϕ_0 is exactly the 0 homomorphism. It turns out there is a homeomorphism between $\hat{A} \cup \{0\}$ and $\hat{\tilde{A}}$ which sends $0 \mapsto \phi_0$. In particular, $\hat{\tilde{A}}$ is (also) the one-point compactification of $\hat{A}$.

Proposition 2.19. *Suppose A is a non-unital commutative* C*-*algebra, and let $\phi_0 : \tilde{A} \to \tilde{A}/A = \mathbb{C}$. The map $\Psi : \hat{A} \cup \{0\} \to \hat{\tilde{A}}$ given by*

$$\Psi(\phi)((a, \lambda)) = \phi(a) + \lambda, \ \forall \ (a, \lambda) \in \tilde{A}$$

is a homeomorphism which sends $0 \mapsto \phi_0$.

Proof. Let $\phi \in \hat{A} \cup \{0\}$. It follows from Exercise 1.23 that $\Psi(\phi) \in \hat{\tilde{A}}$ is the unique extension of ϕ to a character on $\tilde{A}$, meaning that for any $\rho \in \hat{\tilde{A}}$, if $\rho|_A = \phi$, then $\rho = \Psi(\phi)$. In particular, that means $\Psi(0) = \phi_0$. That means we have a bijective correspondence

$$\begin{aligned} \hat{\tilde{A}} &\longleftrightarrow \hat{A} \cup \{0\} \\ \rho &\longrightarrow \rho|_A \\ \Psi(\phi) &\longleftarrow \phi \end{aligned}$$

In particular, $\rho \mapsto \rho|_A$ is the inverse of Ψ. Since $\hat{\tilde{A}}$ and $\hat{A} \cup \{0\}$ are compact, we need only show that Ψ is continuous with respect to the weak*-topology. So, assume $\phi \in \hat{A} \cup \{0\}$ and $\{\phi_i\} \subset \hat{A} \cup \{0\}$ is a net so that $|\phi_i(a) - \phi(a)| \to 0$ for all $a \in A$. Then for any $a \in A$ and $\lambda \in \mathbb{C}$, we have

$$|\Psi(\phi_i)(a + \lambda 1) - \Psi(\phi)(a + \lambda 1)| = |\phi_i(a) + \lambda - (\phi(a) + \lambda)| = |\phi_i(a) - \phi(a)| \to 0.$$

So Ψ is a homeomorphism. □

We can now combine Lemma 2.18 with Proposition 2.19 to see that the bijection of Lemma 2.18 extends to the same bijection $J \mapsto \phi_J$ between maximal ideals in a non-unital A, and characters $\phi \neq \phi_0$ in $\hat{A}$.

Exercise 2.20. Write down the details of the bijection in the non-unital case.

The next key step in proving Theorem 2.11 is the following result; it will later enable us to prove that the Gelfand transform is isometric.

Lemma 2.21. *Let A be a unital* C*-*algebra. For any $a \in A$,*

$$\sigma(a) = \sigma(\Gamma(a)) = \{\phi(a) \ : \ \phi \in \hat{A}\} = ran(\Gamma(a)),$$

and Γ is isometric.

Proof. First, we show that $a \in A$ is invertible iff $\Gamma(a) \in C(\hat{A})$ is invertible. The forward direction follows immediately from the fact that Γ is a homomorphism (Exercise 2.13). On the other hand, if $a \in A$ is not invertible, then a lives in some maximal ideal, meaning $a \in \ker \phi$ for some nonzero character $\phi \in \hat{A}$. Then $\Gamma(a)(\phi) = \phi(a) = 0$, meaning $\Gamma(a)$ is not invertible.

Replacing a with $\lambda 1 - a$, we see that $\sigma(a) = \sigma(\Gamma(a))$ for all $a \in A$. That is, $\lambda \in \sigma(a)$ iff there exists $\phi \in \hat{A}$ such that $\Gamma(\lambda 1 - a)(\phi) = 0$, i.e. $\Gamma(a)(\phi) = \lambda$.

Since A is commutative, every element in A is normal. It follows from Lemma 1.28 that $\|\Gamma(a)\|_\infty = r(a) = \|a\|$ for any $a \in A$. In other words, Γ is isometric. □

Remark 2.22. Notice that the above argument shows that when A is not unital, its Gelfand transform extends to the Gelfand transform on its unitization.

Lemma 2.23. *Let A be a commutative* C*-*algebra. If $a \in A$ is self-adjoint, then $\sigma(a) \subset \mathbb{R}$.*

Proof. Suppose $a \in A$ is self-adjoint, and view $A \subset \tilde{A}$. For each $t \in \mathbb{R}$, the power series

$$\sum_{n \in \mathbb{N}_0} \frac{(ita)^n}{n!}$$

converges to some element in $\tilde{A}$, which we will suggestively call $\exp(ita)$. (**Exercise:** Why do we need to work in the unitization here?) One checks that

$$\exp(ita)^* = \sum_{n \in \mathbb{N}} \frac{(-ita)^n}{n!} = \exp(-ita) = \exp(ita)^{-1},$$

which means $\exp(ita)$ is a unitary in $\tilde{A}$. Now, consider the Gelfand map $\Gamma : \tilde{A} \to C(\hat{\tilde{A}})$. By Lemma 2.21, we know $\sigma(a) = \text{ran}(\Gamma(a)) = \{\phi(a) : \phi \in \hat{A}\}$. So, it suffices to show that $\phi(a) \in \mathbb{R}$ for each $\phi \in \hat{A}$. Fix $\phi \in \hat{A}$. Since ϕ is a character (i.e. continuous, linear, multiplicative), it follows that for any $t \in \mathbb{R}$,

$$\phi(\exp(ita)) = \phi\left(\sum_{n \in \mathbb{N}_0} \frac{(ita)^n}{n!}\right) = \sum_{n \in \mathbb{N}_0} \frac{(it\phi(a))^n}{n!} = e^{it\phi(a)}.$$

Since $\exp(ita)$ is a unitary, we know from Example 1.12 that $e^{it\phi(a)} \in \mathbb{T}$ for all $t \in \mathbb{R}$. Therefore $\phi(a) = \Gamma(a)(\phi) \in \mathbb{R}$ for all characters ϕ and all $a \in A$, and we conclude that $\sigma(a) \subseteq \mathbb{R}$ as desired. □

Remark 2.24. We shall see soon that we did not need to assume A was commutative in Lemma 2.23. The same argument would work by just considering the Gelfand transform on $C^*(a, 1)$. However, we will need to first establish that the spectrum of a in $C^*(a, 1)$ is the same as its spectrum in $\tilde{A}$.

Now we are ready to prove Theorem 2.11.

Proof of the Gelfand–Naimark Theorem. First, we assume that A is unital. We know from Lemma 2.21 that Γ is isometric, which means its image in $C(\hat{A})$ is closed.

For any self-adjoint $a \in A$, we established in Lemma 2.23 that $ran(\Gamma(a)) \subset \mathbb{R}$, which means $\Gamma(a) = \overline{\Gamma(a)}$ is self-adjoint. So Proposition 1.7 tells us Γ is $*$-preserving.

So, invoking Lemma 2.21, $\Gamma(A)$ is a unital, norm closed self-adjoint subalgebra of $C(\hat{A})$ where $\hat{A}$ is compact and Hausdorff. Then the Stone-Weierstrass Theorem ([6, I.5,6]) says that $\Gamma(A) = C(\hat{A})$ provided that it separates the points of $\hat{A}$. But if ϕ and ψ are distinct points in $\hat{A}$, then Lemma 2.18 implies that they have distinct kernels, and so $\Gamma(A)$ separates the points of $\hat{A}$.

Now suppose that A is not unital. Then we know from Remark 2.22 that the Gelfand transform Γ on A extends to the isometric $*$-isomorphism $\tilde{\Gamma} : \tilde{A} \to C(\hat{\tilde{A}})$. Since $A \subseteq \tilde{A}$ is an ideal of co-dimension one, $\tilde{\Gamma}(A)$ is a maximal ideal in $C(\hat{\tilde{A}})$ contained in the maximal ideal $\{f \in C(\hat{\tilde{A}}) : f(\phi_0) = 0\}$. Consequently, $\tilde{\Gamma}(A) = \{f \in C(\hat{\tilde{A}}) : f(\phi_0) = 0\}$, and Proposition 2.19 implies that $\Gamma(A) = C_0(\hat{A})$. □

Corollary 2.25. *Characters on commutative* C*-*algebras are* $*$-*homomorphisms.*

Proof. By Proposition 1.7, it suffices to prove that they map self-adjoint elements to real numbers. For any $a \in A$ self-adjoint, we know from Lemma 2.21 and Lemma 2.23 that

$$\mathbb{R} \supseteq \sigma(a) = \sigma(\Gamma(a)) = \operatorname{ran}(\Gamma(a)) = \{\Gamma(a)(\phi) : \phi \in \hat{A}\} = \{\phi(a) : \phi \in \hat{A}\}.$$ □

Exercise 2.26 (Functoriality of the Gelfand Transform)**.** Suppose that X and Y are two compact topological spaces. Show that a continuous function $\Omega : X \to Y$ induces a *-homomorphism $\Omega_* : C(Y) \to C(X)$ given by $\Omega_*(f) = f \circ \Omega$ for all $f \in C(Y)$. Show that if Ω is a homeomorphism, then Ω_* is a $*$-isomorphism.

Conversely, show that if $\pi : A \to B$ is a unital $*$-homomorphism of (unital) C*-algebras, the function $\hat{\pi} : \hat{B} \to \hat{A}$ given by $\hat{\pi}(\phi)(a) = \phi(\pi(a))$ is continuous, and $\Omega \circ \hat{\pi} = id$.

2.3. **Functional Calculus.** For any element a in a C*-algebra A, we write $\mathrm{C}^*(a)$ for the C*-algebra generated by a. This can be identified as the closure (in the norm inherited from A) of the set of all polynomials in a, a^* with zero constant term, i.e.,

$$\mathrm{C}^*(a) = \overline{\{p(a, a^*) \mid p \in \mathbb{C}[z_1, z_2], p(0,0) = 0\}}.$$

When A is unital, $\mathrm{C}^*(a, 1_A)$ can be identified with the closure of the set of all polynomials on a, a^* (a.k.a. $*$-polynomials on a).[4]

When a is normal, $B := \mathrm{C}^*(a)$ is a commutative C*-algebra, and so we have a $*$-isomorphic Gelfand transform $\Gamma : \tilde{B} \to C(\hat{\tilde{B}})$, which sends B to $C_0(\hat{B}) \subset C(\hat{\tilde{B}})$. Moreover, the multiplicativity of characters implies that any character $\phi \in \hat{B}$ is determined by where it maps a. So, the map $\hat{\tilde{B}} \to \mathbb{C}$ given by $\phi \mapsto \phi(a)$ is a homeomorphism onto $\Gamma(a)(\hat{\tilde{B}})$, which we know by Lemma 2.21 is equal to $\sigma(a)$. In fact, (**exercise**) any net $\{\phi_\lambda\}_\lambda$ in $\hat{\tilde{B}}$ converges wk* to $\phi \in \hat{\tilde{B}}$ iff $\phi_\lambda(a) \to \phi(a)$. The inverse of this homeomorphism, call it $\Omega : \sigma(a) \to \hat{\tilde{B}}$, induces a $*$-isomorphism $\Omega_* : C(\hat{\tilde{B}}) \to C(\sigma(a))$ as in Exercise 2.3: that is,

$$\Omega_*(f)(\phi(a)) := f(\Omega(\phi(a))) = f(\phi), \ \forall\ f \in C(\hat{\tilde{B}}), \phi \in \hat{\tilde{B}}.$$

In particular,

$$\Omega_*(\Gamma(a))(\phi(a)) = \Gamma(a)(\Omega(\phi(a))) = \Gamma(a)(\phi) = \phi(a), \ \forall\ \phi \in \hat{\tilde{B}}. \tag{2.1}$$

Hence we have *-isomorphisms

$$\tilde{B} \xrightarrow{\Gamma} C(\hat{\tilde{B}}) \xrightarrow{\Omega_*} C(\sigma(a)),$$

which takes

$$\mathrm{C}^*(a) = B \xrightarrow{\Gamma} C_0(\hat{B}) \xrightarrow{\Omega_*} C_0(\sigma(a) \backslash \{0\}).$$

Under this composition, what happens to $a \in B$? Well, the generator a of $\mathrm{C}^*(a)$ becomes the function $\Omega_*(\Gamma(a))$, which Equation (2.1) tells us is the function which sends $\phi(a)$ to $\phi(a)$. In other words, **the Gelfand map identifies $a \in \mathrm{C}^*(a)$ with the identity function $z \mapsto z$ on $C(\sigma(a))$.**

When a is not invertible, $C_0(\hat{B})$ corresponds to the ideal consisting of functions that vanish at 0. If a is invertible, then $0 \notin \sigma(a)$, so either way, we can say

$$\mathrm{C}^*(a) \cong C_0(\sigma(a) \backslash \{0\}).$$

[4]If 1_A can be expressed via a *-polynomial in a, e.g., if a is a unitary, then we have $\mathrm{C}^*(a) = \mathrm{C}^*(a, 1_A)$.

Moreover the Gelfand map gives the following identifications:

$$\begin{aligned} a &\longleftrightarrow (z \mapsto z) \\ a^* &\longleftrightarrow (z \mapsto \bar{z}) \\ (1_A &\longleftrightarrow (z \mapsto 1)) \\ p(a, a^*) &\longleftrightarrow p(z, z^*) \end{aligned}$$

where p is a polynomial in two variables with $\mathbb{C}$ coefficients (and no nonzero constant term if we are in the non-unital setting). **Exercise:** Convince yourself that these statements are true.

Problem: What do we mean by $\sigma(a)$ here? By design, this must be the set of $\lambda \in \mathbb{C}$ such that $\lambda 1_A - a$ is not invertible **in B (or** $\mathrm{C}^*(B, 1_A)$, i.e. this is $\sigma_B(a)$, not $\sigma_A(a)$. In general, $\sigma_A(a)$ is smaller because A contains more elements and hence more potential inverses, and we have no reason to suspect that $\sigma_B(a) = \sigma_A(a)$. However, this is in fact true!

Proposition 2.27. *Let a be a normal element of a* C^**-algebra A and $B = \mathrm{C}^*(a)$. Then $\sigma_A(a) = \sigma_B(a)$.*

Proof. Since there are more candidates for an inverse in A than in B, we know immediately that $\sigma_A(a) \subset \sigma_B(a)$. Suppose $\lambda \in \sigma_B(a)$. Then for each $\varepsilon > 0$, there exists $b \in B$ with $\|\Gamma(b)\| = 1$ and $\|\lambda\Gamma(b) - \Gamma(a)\Gamma(b)\| < \varepsilon$. (For example, we can find some some "bump function" $f \in C(\sigma_B(a))$ so that $0 \le f \le 1$, $f(\lambda) = 1$ and $\mathrm{supp}(f) \subset B_{\varepsilon/2}(\lambda)$. Then $b = \Gamma^{-1}(f)$.) That means $\|b\| = 1$ and $\|\lambda b - ab\| < \varepsilon$, which means $\lambda 1_A - a$ is not invertible in $\tilde{A}$. (Indeed, if $c \in A$ such that $c(\lambda 1_A - a) = 1$, then $1 = \|b\| = \|c(\lambda 1_A - a)b\| < \|c\|\varepsilon$ for all ε.) □

This justifies the terminology "spectrum" for the space of characters on a commutative C^*-algebra.

Before moving too far away from Proposition 2.27, we remark that it yields a more general corollary.

Corollary 2.28. *If a is a normal element in a unital* C^**-algebra A and B is any unital* C^**-subalgebra of A containing a, then $\sigma_A(a) = \sigma_B(a)$.*

Remark 2.29. Using a holomorphic functional calculus argument, one can use a similar argument (with approximating polynomials) to show the same holds for non-normal elements.

Now we come to an incredibly powerful tool, with which we conclude the section: *The Functional Calculus.* Let A be a unital C^*-algebra, $a \in A$ a normal element, and $f \in C(\sigma(a))$. We denote by $f(a)$ the inverse image of f under the Gelfand transform of $\mathrm{C}^*(a, 1)$ (the isometric $*$-isomorphism between $\mathrm{C}^*(a, 1)$ and $C(\sigma(a))$).

Corollary 2.30 (The Functional Calculus)**.** *Let a be a normal element of a unital* C^**-algebra A. Then for any $f \in C(\sigma(a))$, $f(a) \in A$ is normal, and for any $g \in C(\sigma(f(a)))$ we have the following:*

(1) $f(\sigma(a)) = \sigma(f(a))$,
(2) $g(f(\sigma(a)) = (g \circ f)(a)$, and
(3) if $0 \in \sigma(a)$ and $f(0) = 0$, then $f(a)$ is in the non-unital C^**-algebra,* $\mathrm{C}^*(a)$.

Proof. To see that $f(a)$ is normal, use the Stone-Weierstrass Theorem to approximate $f \in C(\sigma(a))$ by a sequence of Laurent polynomials; the fact that polynomials in a are normal means that $f(a)$ is too. (**Exercise:** Check the details!)

Since $f(a) \in \mathrm{C}^*(a, 1_A)$, we have

$$\sigma(f(a)) = \sigma(\Gamma(f(a))) = \sigma(f) = f(\sigma(a)).$$

Since Γ is a homomorphism, the second claim holds immediately when g is a Laurent polynomial (i.e. a polynomial in z and $\overline{z}$). Then the general case follows by approximating g uniformly with Laurent polynomials (again using Stone-Weierstrass).

The third claim follows immediately from Proposition 2.27. □

Remark 2.31. Suppose $a \in A$ is a normal element of a C^*-algebra and $X \subset \mathbb{C}$ is a subspace so that $\sigma(a) \subset X$. Then any continuous function on X restricts to one on the compact subset $\sigma(a)$, i.e., if $f \in C_0(X)$, then $f|_{\sigma(a)} \in C(\sigma(a))$. The upshot is that when we want to define some $f \in C(\sigma(a))$, we usually just define a continuous f on the ball of radius $\|a\|$ about the origin, or even on $[-\|a\|, \|a\|]$ when a is self-adjoint.

Exercise 2.32. Suppose $a \in A$ is a normal *invertible* element in a unital C^*-algebra and $\Gamma : \mathrm{C}^*(a) \to C_0(\sigma(a)\backslash\{0\})$ is the Gelfand transform. Must we have $a^{-1} \in \mathrm{C}^*(a)$?

We will see this applied repeatedly in Chapter 3.

Exercise 2.33. Suppose A and B are commutative unital C*-algebras and $\phi : A \to B$ a unital $*$-homomorphism. Then for any $a \in A$ and $f \in C(\sigma(a))$, we have $\phi(f(a)) = f(\phi(a))$.

Exercise 2.34. Let $\pi : A \to B$ be a surjective $*$-homomorphism between C*-algebras and $b \in B$ a self-adjoint element. Show that b lifts to a self-adjoint element $a \in A$ with $\pi(a) = b$ and $\|a\| = \|b\|$.

Exercise 2.35. Suppose $A = C_0(X)$. Write down an explicit formula for the Gelfand transform $\Gamma : A \to C_0(\hat{A})$ in this case.

3. Positive elements

Preview of Lecture:

- Exercise 3.3 is a cornerstone of the theory of C*-algebras; see if you can figure out why it's true before lecture!
- In lecture, we will prove Proposition 3.6 and Example 3.9.
- We will not prove Corollary 3.7 in lecture.
- Theorem 3.10 is really important, and the proof uses all of the exercises that precede it in this section, but it's otherwise pretty straightforward. We won't discuss the proof.
- We will discuss the proofs of Proposition 3.15 and Corollary 3.16 in lecture.

The Functional Calculus is an incredibly powerful tool for handling normal elements. Of course, not every element in a C*-algebra is normal. Nonetheless, by associating to each element $a \in A$ the self-adjoint element $a^*a \in A$, we can spread the influence of the functional calculus to an entire non-commutative C*-algebra. It turns out that elements of the form a^*a take on an even more important structural role in C*-algebras, which we will explore now.

Definition 3.1. A self-adjoint element a in a C*-algebra A is *positive* if $\sigma(a) \subset [0, \infty)$. We denote this by $a \geq 0$.

This allows us to define a partial ordering on the self-adjoint elements of A: for a and b self-adjoint, define $a \leq b$ if $b - a \geq 0$.

Example 3.2. The positive elements in $C_0((0, 1])$ are exactly the ones whose range (i.e. spectrum) lies in $[0, \infty)$.

Let's start with a few observations using the functional calculus:

Exercise 3.3. Each positive element in a C*-algebra has a unique positive square root.

Exercise 3.4. If $a \in A$ is a self-adjoint element, then there exist positive elements a_+ and a_- such that $a = a_+ - a_-$ and $a_+a_- = a_-a_+ = 0$.

Exercise 3.5. Let $a \in A$ be self-adjoint, a_+ and a_- its positive and negative parts as in Exercise 3.4, and $(a_+)^{1/2}$ and $(a_-)^{1/2}$ their respective unique positive square roots. Show that $a_+(a_-)^{1/2} = 0$ and $(a_+)^{1/2}(a_-)^{1/2} = 0$.

The following proposition is mostly technically useful, but it also showcases some techniques using the functional calculus.

Proposition 3.6. *Let a be a self-adjoint element in a unital C*-algebra A. Then the following are equivalent.*

(1) $a \geq 0$;
(2) $a = b^2$ *for some self-adjoint* $b \in A$;
(3) $\|\alpha 1_A - a\| \leq \alpha$ *for all* $\alpha \geq \|a\|$;
(4) $\|\alpha 1_A - a\| \leq \alpha$ *for some* $\alpha \geq \|a\|$.

Proof. We assume A is unital or pass to its unitization.

That (1) $\Rightarrow$ (2) follows from the functional calculus, and that (3) $\Rightarrow$ (4) is clear.

Assume (2). Let $f \in C(\sigma(b))$ be given by $f(z) = z^2$. Then

$$\|f\|_{\sup} = \|b^2\| = \|a\|,$$

and so (since $\sigma(b) \subseteq \mathbb{R}$ by Lemma 2.23) $0 \leq f \leq \|a\|$. Then $0 \leq \alpha - f \leq \alpha$ for any $\alpha \geq \|a\|$. Then (identifying α with the constant function on $\sigma(b)$ when appropriate), we compute

$$\|\alpha 1_A - a\| = \|\alpha(b) - f(b)\| = \|(\alpha - f)(b)\| = \|\alpha - f\|_{\sup} \leq \alpha.$$

It remains to show (4) $\Rightarrow$ (1). Suppose $\alpha \geq \|a\|$ is such that $\|\alpha 1_A - a\| \leq \alpha$. Let $h(z) = z$ denote the identity function on $\sigma(a)$. Then we have

$$\alpha \geq \|\alpha 1_A - a\| = \|(\alpha - h)(a)\| = \|\alpha - h\|_{\sup} = \sup_{\lambda \in \sigma(a)} |\alpha - \lambda|.$$

It follows that $\sigma(a) \subset [0, \infty)$. Since a was assumed to be self-adjoint, this means $a \geq 0$. $\square$

Some concluding notation: The collection of positive elements in a C*-algebra A is denoted by A_+, and the self-adjoints are often denoted by $A_{s.a.}$

Corollary 3.7. *For a* C*-*algebra A, the sets $A_{s.a.}$ and A_+ are both closed.*

Proof. Suppose x_n is a sequence in $A_{s.a.}$ converging to $x \in A$. Then

$$\|x_n^* - x^*\| = \|x_n - x\| \to 0,$$

and so $x_n = x_n^* \to x^*$. Hence $x^* = x$. Now, suppose $(a_n) \in A_+$ converges to $a \in A$. Then we know $a = a^*$ and $\|a_n\| \to \|a\|$. Assume A is unital or unitize. Let $\alpha = \sup_n \|a_n\| \geq \|a\|$. Then $\alpha 1_A - a_n \to \alpha 1_A - a$, and $\|\alpha 1_A - a_n\| \leq \alpha$ for all n by Proposition 3.6. It follows that $\|\alpha 1_A - a\| \leq \alpha$, which again by Proposition 3.6 implies that a is positive. □

Exercise 3.8. If $a, b \in A$ are positive, then so is $a + b$. If a and b moreover commute, then $ab \geq 0$. Can you think of two positive elements in a C*-algebra whose product is not positive? (Hint: For the first part, you can assume A is unital or work in $\tilde{A}$ (why?). Then use Proposition 3.6. For the question, consider operators in $M_2(\mathbb{C})$.)

Proposition 3.9. *The positive operators in $B(\mathcal{H})$ are exactly the positive semi-definite operators, i.e., $T \in B(\mathcal{H})$ is positive iff $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$.*

Proof. Suppose $T \in B(\mathcal{H})$. By the Proposition 3.6, if $T \geq 0$, then there exists a self-adjoint $S \in B(\mathcal{H})$ such that $T = S^2 = S^*S$. Then for any $x \in \mathcal{H}$, we have

$$\langle Tx, x\rangle = \langle S^*Sx, x\rangle = \langle Sx, Sx\rangle = \|Sx\|^2 \geq 0.$$

Now, suppose $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$. By Exercise 1.56 from the Prereqs, $T = T^*$ and so $\sigma(T) \subset \mathbb{R}$. So, given $\lambda < 0$ we want to show that $\lambda I - T$ is invertible. If $\lambda < 0$, then $-\lambda = |\lambda|$ and so for every nonzero $x \in \mathcal{H}$,

$$\begin{aligned}\|(\lambda I - T)x\|^2 &= |\langle(\lambda I - T)x, (\lambda I - T)x\rangle| \\ &= |\|Tx\|^2 + 2|\lambda|\langle Tx, x\rangle + |\lambda|^2\|x\|^2| \\ &= \|Tx\|^2 + 2|\lambda|\langle Tx, x\rangle + |\lambda|^2\|x\|^2 \\ &\geq |\lambda|^2\|x\|^2.\end{aligned}$$

That means that for every $x \in \mathcal{H}$, $\|(\lambda I - T)x\| \geq |\lambda|\|x\|$. In other words, the operator $\lambda I - T$ is bounded below, which means it is injective (Exercise 1.60 from Prereqs). So, by the Open Mapping Theorem, to show that $\lambda I - T$ is invertible, it remains to show that it is surjective.

For any operator $S \in B(\mathcal{H})$, $\ker(S) = (S^*(\mathcal{H}))^\perp$ (Exercise 1.58 from Prereqs). Since $(\lambda I - T) = (\lambda I - T)^*$, the above argument shows that $\ker(\lambda I - T) = 0 = ((\lambda I - T)(\mathcal{H}))^\perp$, which means $T - \lambda$ is surjective and thus invertible. □

Theorem 3.10. *For any $a \in A$, the element a^*a is positive.*

Proof. Suppose $b = a^*a \in A$. Then b is self-adjoint, and hence by Exercise 3.4, we can write it as $b = b_+ - b_-$ for some $b_+, b_- \geq 0$ with $b_+b_- = 0$. We want to show that $b_- = 0$. Since b_- is self-adjoint, we know $\|b_-\| = r(\sigma(b_-))$, and so it suffices to show that $\sigma(b_-) = \{0\}$. Now, for notational ease, we write $c = a(b_-)^{1/2}$, where $(b_-)^{1/2}$ is its unique positive square root. By Exercise 3.5, we have that $(b_-)^{1/2}b_+ = 0$, and so we compute

$$-c^*c = -(b_-)^{1/2}a^*a(b_-)^{1/2} = -(b_-)^{1/2}b(b_-)^{1/2} = -(b_-)^{1/2}(b_+ - b_-)(b_-)^{1/2} = b_-^2.$$

Then $-c^*c = b_-^2 \geq 0$, which means $\sigma(-c^*c) \subset [0, \infty)$.

Write $c = \mathrm{Re}(c) + i\mathrm{Im}(c)$ as in (1.1). Then we compute

$$\begin{aligned}cc^* &= [c^*c + cc^*] - c^*c \\ &= [(\mathrm{Re}(c) + i\mathrm{Im}(c))^*(\mathrm{Re}(c) + i\mathrm{Im}(c)) + (\mathrm{Re}(c) + i\mathrm{Im}(c))(\mathrm{Re}(c) + i\mathrm{Im}(c))^*] - c^*c \\ &= 2(\mathrm{Re}(c)^2 + \mathrm{Im}(c)^2) + b_-^2.\end{aligned}$$

Then cc^* is the sum of positive elements, and hence is positive. Since[5] $\sigma(cc^*)\cup\{0\} = \sigma(c^*c)\cup\{0\}$, it follows that both cc^* and c^*c have non-negative spectra. Since $\sigma(-c^*c) = -\sigma(c^*c)$, it follows that $\sigma(c^*c) = \{0\}$. Since c^*c is self-adjoint, its norm is its spectral radius, and so

$$0 = \|c^*c\| = \|-c^*c\| = \|b_-^2\| = \|b_-\|^2,$$

and we are done. □

Exercise 3.11. Let A be a C*-algebra. Show the following:

(1) If $a, b \in A$ are self-adjoint with $a \le b$ and $c \in A$, then $c^*ac \le c^*bc$. (Hint: Take a square root and use the previous theorem.)
(2) Assuming A is a unital C*-algebra and $a \in A$ positive, show that $a \le \|a\|1$. Moreover, $\|a\| \le 1$ iff $a \le 1$. In this case we also have $1 - a \le 1$ and $\|1 - a\| \le 1$.
(3) If A is unital and $a \in A$ is invertible, then so is a^*, a^*a, and $(a^*a)^{1/2}$. Moreover, the inverses are in $C^*(a)$.

Exercise 3.12. Let $\pi : A \to B$ be a surjective $*$-homomorphism between C*-algebras and $b \in B$ a positive element. Show that b lifts to a positive element $a \in A$ (that is, there is $a \in A_+$ with $\pi(a) = b$) such that $\|a\| = \|b\|$. (Hint: How would you modify the proof for self-adjoint lifts? Sketch it.)

Exercise 3.13. Show that if $0 \le a \le b$ then $0 \le \|a\| \le \|b\|$. What happens if we drop the assumption that $0 \le a$?

Exercise 3.14. Suppose $0 \le a \le b$ and $a, b \in GL(A)$ are invertible. Prove that $0 \le b^{-1} \le a^{-1}$. (Try not to assume that a, b commute – that makes the exercise much easier.)

3.1. **Polar decomposition.** For each $a \in A$, denote by $|a|$ the unique positive square root of a^*a, i.e.

$$|a| = (a^*a)^{1/2}.$$

Proposition 3.15. *For each operator $T \in B(\mathcal{H})$, there is a unique partial isometry $U \in B(\mathcal{H})$ with* $\ker(U) = \ker(T)$ *and $U|T| = T$. Moreover $|T| \in C^*(T)$ and $U \in C^*(T)''$. If T is invertible, then U is a unitary.*

The description $T = U|T|$ is called the *polar decomposition* of T, in analogy with the fact that every complex number z can be written as a norm-1 element e^{it}, times a non-negative real number r. U is sometimes called the *polar part* of T and $|T|$ is the *positive part*.

Proof. Note that for all $\xi \in H$, we have

$$\|T\xi\|^2 = \langle T\xi, T\xi\rangle = \langle T^*T\xi, \xi\rangle = \langle |T|^2\xi, \xi\rangle = \langle |T|\xi, |T|\xi\rangle = \||T|\xi\|^2. \tag{3.1}$$

It follows that the linear map $U_0 : |T|\mathcal{H} \to T\mathcal{H}$ given by $|T|x \mapsto Tx$ is isometric, and hence extends to an isometry $\overline{|T|\mathcal{H}} \to \overline{T\mathcal{H}}$ (also denoted U_0). We define $U \in B(\mathcal{H})$ to be U_0 on $\overline{|T|\mathcal{H}}$ and 0 on $(|T|\mathcal{H})^\perp$. It follows from Exercise 1.53 from the Prereqs that U is a partial isometry with $U^*|_{\overline{T\mathcal{H}}} = U_0^{-1}$ and $\ker(U^*) = (T\mathcal{H})^\perp$, and by definition $U|T| = T$. Moreover, we have from (3.1) and Exercise 1.58 from the Prereqs that $\ker(U) = |T|(\mathcal{H})^\perp = \ker(|T|) = \ker(T)$.

For uniqueness, suppose $V \in B(\mathcal{H})$ is another partial isometry with $\ker(V) = \ker(T)$ and $V|T| = T$. Since $V|_{|T|\mathcal{H}} = U|_{|T|\mathcal{H}}$, it follows from continuity that they also agree on $\overline{|T|\mathcal{H}}$. As $\ker(V) = \ker(T) = \ker(U) = (|T|(\mathcal{H}))^\perp$ by construction, the fact that $\mathcal{H} = \overline{|T|\mathcal{H}} \oplus (|T|\mathcal{H})^\perp$ implies via a linearity argument that $V\xi = U\xi$ for any $\xi \in \mathcal{H}$.

It follows from the functional calculus that $|T| \in C^*(T^*T) \subset C^*(T)$. Now, suppose $S \in C^*(T)'$. If $\xi \in \ker(T) = \ker(U)$, then $TS\xi = ST\xi = 0$ and so $S\xi \in \ker(T) = \ker(U)$. Then $US\xi = 0 = SU\xi$ for every $\xi \in \ker(T) = (|T|\mathcal{H})^\perp$. For $\xi = |T|\eta \in |T|\mathcal{H}$, we have

$$US\xi = US|T|\eta = U|T|S\eta = TS\eta = ST\eta = SU|T|\eta = SU\xi.$$

Since $|T|\mathcal{H}$ is dense in $\overline{|T|\mathcal{H}}$, it follows that $US = SU$ on $\overline{|T|\mathcal{H}}$ and on $(|T|\mathcal{H})^\perp$. Then it follows by a linearity argument as above that S and U commute. Hence $U \in C^*(T)''$.

[5]This is a more general ring theoretic fact that $\sigma(xy)\cup\{0\} = \sigma(yx)\cup\{0\}$ for any x, y in a complex unital ring. Indeed, if $0 \ne \lambda \notin \sigma(xy)$, then there exists z such that $z(\lambda - xy) = 1 = (\lambda - xy)z$. Then $\lambda^{-1}(\lambda + yzx)$ is the inverse of $\lambda - yx$. Check this if you haven't seen it before!

Finally, if T is invertible, then by Exercise 3.11 (3), so is $(T^*T)^{1/2}$. Then we have

$$U = T(T^*T)^{-1/2},$$

and one checks that $U^*U = UU^* = I$. $\square$

As the range space of U is $\overline{T\mathcal{H}}$, and U is a partial isometry, it follows that $UU^* = \text{proj}_{\overline{T\mathcal{H}}}$. Similarly, the source projection of U is $U^*U = \text{proj}_{\overline{|T|\mathcal{H}}}$.

Corollary 3.16. *Let $T \in B(\mathcal{H})$ with polar decomposition $T = U|T|$. Then $|T^*| = U|T|U^*$ and $T^* = U^*|T^*|$.*

Proof. We know from Exercise 3.11 (2) that $U|T|U^*$ is positive, and since $U^*U = \text{proj}_{\overline{|T|\mathcal{H}}}$ and $T^* = (U|T|)^* = |T|U^*$,

$$(U|T|U^*)(U|T|U^*) = U|T|^2U^* = TT^*.$$

By the uniqueness of the square root, we have $U|T|U^* = (TT^*)^{1/2} = |T^*|$. From this we further deduce

$$U^*|T^*| = U^*U|T|U^* = |T|U^* = (U|T|)^* = T^*.$$ $\square$

Exercise 3.17. Where possible, give geometric (or "spatial") as well as algebraic explanations for the following statements about the polar decomposition:

(1) $U^*U|T| = |T|$,
(2) $U^*T = |T|$, and
(3) $UU^*T = T$.

Exercise 3.18. Show that T is compact iff $|T|$ is compact.

In general, for $T \in B(\mathcal{H})$, the partial isometry U in the polar decomposition $T = U|T|$ is not in $\mathrm{C}^*(T)$. However, it turns out that if you take any continuous function $f \in C(\sigma(|T|)\backslash\{0\})$, the operator $Uf(|T|)$ is in $\mathrm{C}^*(T)$.

Proposition 3.19. *Let $T \in B(\mathcal{H})$ with polar decomposition $T = U|T|$, and $f \in C_0(\sigma(|T|)\backslash\{0\})$. Then $Uf(|T|) \in \mathrm{C}^*(T)$. Moreover, $U^*US = S$ for all $S \in \mathrm{C}^*(T)$.*

Proof. Again by Stone-Weierstraß, any $f \in C(\sigma(|T|))$ is the norm limit of polynomials. Moreover, if $f(0) = 0$, then we can assume the same for an approximating sequence of polynomials. (In other words $f \in C_0(\sigma(|T|)\backslash\{0\}$ can be approximated by polynomials in $C_0(\sigma(|T|)\backslash\{0\})$. Note that these are polynomials with zero constant term, i.e. $p(0) = 0$.) So, if the claim holds for all polynomials p with $p(0) = 0$, it holds for any $f \in C_0(\sigma(|T|)\backslash\{0\})$. Let $p(z) = \sum_{k=1}^{n} \lambda_k z^k$. Then since $|T| \in \mathrm{C}^*(T)$, we have

$$Up(|T|) = \sum_{k=1}^{n} \lambda_k U|T|^k = \sum_{k=1}^{n} \lambda_k T|T|^{k-1} \in \mathrm{C}^*(T).$$

$\square$

4. Ideals, Approximate Units, and $*$-homomorphisms

Preview of Lecture: To help guide your reading, we indicate here which of the following material we will address in lecture and which we will assume familiarity with:

The lecture for this section will focus on Theorem 4.13. I'd encourage you to read the proof of Theorem 4.4 on your own; it's a lovely application of the functional calculus and is just a bit too long to fit into lecture.

The techniques in the proofs of Lemma 4.8 and Theorem 4.11 do not translate well to lecture, but that does not detract from their importance. In fact, they showcase a powerful yet technical tool: an approximate unit (a.k.a. approximate identity). Many C*-algebraists (guilty!) are intimidated by these at first. But the first time you use them in your own research, you'll love them for life.

Definition 4.1. An *approximate identity* in a C*-algebra A is an increasing net $(e_\lambda)_{\lambda\in\Lambda}$ of positive contractive elements (i.e. $0 \le e_\gamma \le e_\lambda$ and $\|e_\lambda\| \le 1$ for all $\lambda, \gamma \in \Lambda$ with $\lambda \ge \gamma$) such that

$$\lim_\lambda \|e_\lambda a - a\| = 0.$$

Exercise 4.2. Prove that if $(e_\lambda)_{\lambda\in\Lambda}$ is an approximate identity, then for any $a \in A$ we also have

$$0 = \lim_\lambda \|a e_\lambda - a\| = \lim_\lambda \|e_\lambda a e_\lambda - a\|.$$

Exercise 4.3. Suppose A is a C*-algebra with approximate identity $(e_\lambda)_{\lambda\in\Lambda}$. Show that for any $a \in A$, we have

$$\lim_\lambda \|a e_\lambda - a\| = \lim_\lambda \|e_\lambda a e_\lambda - a\| = \lim_\lambda \|e_\lambda^2 a e_\lambda^2 - a\| = \lim_\lambda \|e_\lambda^{1/2} a e_\lambda^{1/2} - a\| = 0.$$

(Hint: For the last part, remember that on $[0, 1]$, the function $x \mapsto x^{1/2}$ can be uniformly approximated by polynomials.)

Theorem 4.4. *Every* C**-algebra has an approximate identity. In fact, if $J \lhd A$ is an ideal in a* C**-algebra A, then there exists an approximate identity for J. Moreover, every separable* C**-algebra has a separable approximate identity.*

The proof we give here (which we found in unpublished lecture notes of Dana Williams) relies heavily on the functional calculus.

Before we prove the theorem, though, here are a few exercises to build intuition.

Exercise 4.5. Prove that in $K(\ell^2)$, the projections

$$P_n : (\xi_n)_n \mapsto (\xi_1, \dots, \xi_n, 0, 0, \dots)$$

form an approximate identity. (Be careful here; you can't assume that a given finite rank operator looks like a finite matrix when expressed in the basis $\{e_i\}_{i\in\mathbb{N}}$! But if T is finite rank, pick a basis $\{\zeta_i\}_{i=1}^k$ for range(T), and recall that each $\zeta_i \in \ell^2$, to finish the proof.)

For a general Hilbert space, we form the approximate identity for the compact operators by nets of projections with finite rank where the order is given by the natural order on the projections, i.e. $p \le q$ iff $pq = qp = p$.

Exercise 4.6. Determine an approximate identity for $C_0((0, 1])$. (A sketch will do.) Now, suppose A is a C*-algebra and $a \in A$ a positive element with $\|a\| \le 1$. Give an approximate identity for C$^*(a)$.

Proof of Theorem 4.4. If $J \lhd A$ is an ideal, then J is also an ideal in $\tilde{A}$. Thus, we will assume that A is unital.

For each integer $n \in \mathbb{N}$, define $f_n : [0, \infty) \to [0, 1)$ by

$$f_n(t) = \frac{nt}{1 + nt}.$$

By the functional calculus, since f_n is continuous, if $a \in A_+ \cap J$, then

$$f_n(a) := (na)(1 + na)^{-1}$$

is a well-defined element of J.

Let Λ be the set of finite subsets of $J_{s.a.}$, ordered by inclusion. For $\lambda \in \Lambda$, write $\lambda = \{x_1, \dots, x_n\}$, and define

$$e_\lambda := f_n(x_1^2 + \cdots + x_n^2).$$

(Check: Why is $x_1^2 + \cdots + x_n^2 \in A_+ \cap J$?) I claim that $(e_\lambda)_{\lambda\in\Lambda}$ is the desired approximate unit for J.

Exercise: Show that $0 \leq e_\lambda$ and $\|e_\lambda\| \leq 1$ for all λ.

The next step is to check that if $\lambda \leq \gamma$, then $e_\lambda \leq e_\gamma$. However, computationally, it will be easier (and sufficient) to prove that

$$\lambda \leq \gamma \Rightarrow 1_A - e_\gamma \leq 1_A - e_\lambda.$$

Notice that $1 - f_n(t) = \frac{1+nt}{1+nt} - \frac{nt}{1+nt} = \frac{1}{1+nt}$; the functional calculus then implies that if $\lambda = \{x_1, \dots, x_n\}$,

$$1_A - e_\lambda = (1_A + n(x_1^2 + \cdots + x_n^2))^{-1}.$$

So, if $\gamma \geq \lambda$, then since Λ is ordered by inclusion, we must have $\gamma = \{x_1, \dots, x_n, x_{n+1}, \dots, x_m\}$ for some $m \geq n$. Therefore

$$1_A - e_\gamma = (1_A + m(x_1^2 + \cdots + x_m^2))^{-1} = (1_A + n(x_1^2 + \cdots + x_n^2) + (m-n)(x_1^2 + \cdots + x_n^2) + m(x_{n+1}^2 + \cdots + x_m^2))^{-1}.$$

By Exercise 3.14, the fact that $(m-n)(x_1^2 + \cdots + x_n^2) + m(x_{n+1}^2 + \cdots + x_m^2)$ is positive implies that $1_A - e_\gamma \leq 1_A - e_\lambda$, or equivalently, $e_\lambda \leq e_\gamma$ if $\lambda \leq \gamma$.

Finally, we have to prove that $\lim_\lambda \|e_\lambda a - a\| = 0$ for all $a \in J$. To this end, define $g_n(t) : [0,\infty) \to [0,\infty)$ by

$$g_n(t) = (1 - f_n(t))t(1 - f_n(t)) = \frac{t}{(1+nt)^2}.$$

Some first-semester calculus tells us that $g_n(t)$ achieves its maximum at $t = 1/n$; it follows that

$$\|g_n(t)\|_\infty = \frac{1}{4n}.$$

Now, suppose $a \in J_{s.a.}$. Then $\{a\} \in \Lambda$; and for any $\lambda_0 \in \Lambda$, there is $\lambda \in \Lambda$ with $\lambda \geq \lambda_0$ and $\lambda \geq a$. (What is λ, exactly?) So, when we compute $\lim_{\lambda\in\Lambda} \|ae_\lambda - a\|$, we can assume $\lambda \geq \{a\}$. Moreover, notice that

$$\|ae_\lambda - a\|^2 = \|(ae_\lambda - a)^*(ae_\lambda - a)\| = \|e_\lambda a^2 e_\lambda - e_\lambda a^2 - a^2 e_\lambda + a^2\|,$$

and $e_\lambda a^2 e_\lambda - e_\lambda a^2 - a^2 e_\lambda + a^2 = (1_A - e_\lambda)a^2(1_A - e_\lambda)$.

Now, if we assume $\lambda \geq \{a\}$ and write $\lambda = \{x_1, \dots, x_n\}$, then Exercise 3.11(a) implies that

$$(1_A - e_\lambda)a^2(1_A - e_\lambda) \leq (1_A - e_\lambda)(x_1^2 + \cdots + x_n^2)(1_A - e_\lambda) = g_n(x_1^2 + \cdots + x_n^2).$$

Putting it all together, we see that

$$\|ae_\lambda - a\|^2 \leq \|g_n(x_1^2 + \cdots + x_n^2)\| \leq \frac{1}{4n},$$

which tends to 0 as we let the sets λ grow in size. Thus $\|ae_\lambda - a\| \to 0$ for all $a \in J_{s.a.}$.

If $a \in J$ is not self-adjoint, we compute:

$$\|a - ae_\lambda\|^2 = \|a(1-e_\lambda)\|^2 = \|(a(1-e_\lambda))^*(a(1-e_\lambda))\| = \|(1-e_\lambda)a^*a(1-e_\lambda)\| \leq \|1-e_\lambda\|\,\|a^*a - a^*ae_\lambda\|.$$

Since $0 \leq 1_A - e_\lambda \leq 1_A$, Exercise 3.13 implies $\|a - ae_\lambda\|^2 \leq \|a^*a - a^*ae_\lambda\|$, which tends to zero since $a^*a \in J_{s.a.}$.

Finally, if J is separable,[6] so is $J_{s.a.}$. So, let $\{x_i\}_{i\in\mathbb{N}} \subseteq J_{s.a.}$ be a dense sequence. Then we replace Λ in the above proof with $\Gamma = \{\{x_1, \dots, x_n\} : n \in \mathbb{N}\}$; it's an **Exercise** to check that $\{e_\gamma : \gamma \in \Gamma\}$ is an approximate identity, which is evidently countable. □

Remark 4.7. A C*-algebra with a countable approximate identity is called *σ-unital*. Any separable C*-algebra is σ-unital, but there exist non-separable σ-unital C*-algebras. A silly example is $B(\ell^2)$ since it's actually unital; a non-silly example is $C_0(X)$ where X is a locally compact, but not σ-compact, Hausdorff space. Many results that hold in the separable setting can be generalized to the σ-unital setting.

There are a few interesting characterizations of a σ-unital C*-algebra, such as containing a *strictly positive element*, which is an element $h \in A$ such that $\phi(h) > 0$ for every nonzero positive $\phi \in A^*$. For more on this, see [16, Section 3.10] and the following section on hereditary subalgebras.

[6] Recall: a separable topological space is one with a countable dense subset.

Here is a quick application of approximate identities.

Lemma 4.8. *Every closed two-sided ideal in a* C^**-algebra is self-adjoint.*

Proof. Let J be a closed two-sided ideal in A. Then $B = J \cap J^*$ is a C^*-subalgebra of A such that $x^*x, xx^* \in B$ for all $x \in J$. Let (e_λ) be an approximate identity for B. Then for any $x \in J$, we have $xx^* - xx^*e_\lambda \in B$ and hence

$$\begin{aligned}\lim_\lambda \|x^* - x^*e_\lambda\|^2 &= \lim_\lambda \|(x - e_\lambda x)(x^* - x^*e_\lambda)\| \\ &= \lim_\lambda \|(xx^* - xx^*e_\lambda) - e_\lambda(xx^* - xx^*e_\lambda)\| = 0.\end{aligned}$$

Since $x^*e_\lambda \in J$, it follows that $x^* \in J$ and so $J = J^*$. □

This means that every ideal in a C^*-algebra is a C^*-subalgebra, which means that each ideal has an approximate unit. In fact, more is true. A net (a_λ) in a C^*-algebra A is *quasi-central* if $\lim_\lambda \|a_\lambda b - ba_\lambda\| = 0$ for every $b \in A$. We have the following extension of the above theorem ([8, Theorem I.9.16]).

Theorem 4.9. *Every ideal of a* C^**-algebra has a quasi-central approximate unit.*

Exercise 4.10. Suppose A is a C^*-algebra with a closed two-sided ideal $J \triangleleft A$ and a C^*-subalgebra $I \subset A$ such that $I \triangleleft J$. Show that $I \triangleleft A$.

An approximate identity will also enable us to prove that the quotient of any C^*-algebra by a closed two-sided ideal is again a C^*-algebra.

Theorem 4.11. *Let* A *be a* C^**-algebra and* $J \triangleleft A$*. Then* A/J *is a* C^**-algebra.*

Proof. Since $J \subset A$ is a Banach subalgebra, a basic result from functional analysis (cf. [6, Theorems III.4.2 and VII.2.6]) implies that A/J is a Banach algebra under the norm $\|a + J\| = \inf_{x \in J} \|a + x\|$. (**Exercise:** Prove it!) Moreover, from the fact that $\|b\| = \|b^*\|$ for all $b \in A$, a two-line calculation shows that $\|a + J\| = \|a^* + J\|$ for all $a \in A$. So, we just check the C^*-identity for $\|a + J\| = \inf_{x \in J} \|a + x\|$. Let $a \in A$ and (e_λ) an approximate identity for J. First, we claim that $\|a + J\| = \lim_\lambda \|a - ae_\lambda\|$. Since $ae_\lambda \in J$ for each λ, the $\leq$ inequality is clear. For the other direction, let $\varepsilon > 0$ and $x \in J$ such that $\|a + J\| + \varepsilon > \|a - x\|$. By possibly passing to $\tilde{A}$, we assume A is unital. Then by Exercise 3.11, $\|1 - e_\lambda\| \leq 1$, and

$$\begin{aligned}\lim_\lambda \|a - ae_\lambda\| &\leq \lim_\lambda \|(a - x)(1 - e_\lambda)\| + \|x - xe_\lambda\| \\ &\leq \lim_\lambda \|a - x\| + \|x - xe_\lambda\| \\ &= \|a - x\| < \|a + J\| + \varepsilon.\end{aligned}$$

Now, we can check the C^*-norm:

$$\begin{aligned}\|(a + J)^*(a + J)\| &= \|a^*a + J\| = \lim_\lambda \|a^*a(1 - e_\lambda)\| \geq \lim_\lambda \|1 - e_\lambda\|\|a^*a(1 - e_\lambda)\| \\ &\geq \lim_\lambda \|(1 - e_\lambda)aa^*(1 - e_\lambda)\| = \lim_\lambda \|a(1 - e_\lambda)\|^2 = \|a + J\|^2 \\ &= \|a^* + J\|\|a + J\| \geq \|(a + J)^*(a + J)\|.\end{aligned}$$

□

Exercise 4.12. Let $\pi : A \to B$ be a $*$-homomorphism between C^*-algebras. Check that $\ker(\pi)$ is a closed two-sided ideal in A and the quotient map $q : A \to A/\ker(\pi)$ is a $*$-homomorphism.

Now, we are ready to build on Proposition 1.29 to get a very powerful theorem for $*$-homomorphisms.

Theorem 4.13. *An injective* $*$*-homomorphism between* C^**-algebras is isometric. The image of any* $*$*-homomorphism between* C^**-algebras is a* C^**-algebra. In particular, the range of any* $*$*-homomorphism between* C^**-algebras is closed.*

Proof. Recall from Proposition 1.29 that a $*$-homomorphism $\phi : A \to B$ between C^*-algebras is contractive and for any $a \in A$, $\sigma(\phi(a)) \subset \sigma(a)$. We give the proof under the assumption that our C^*-algebras and our maps are all unital and leave the adaption to the non-unital setting as an **Exercise**.

Let $\phi : A \to B$ be an injective $*$-homomorphism. Note that for any $a \in A$, $\|a\|^2 = \|a^*a\|$ and $\|\phi(a)\|^2 = \|\phi(a)^*\phi(a)\| = \|\phi(a^*a)\|$, so by Theorem 3.10, it suffices to prove that $\|\phi(a)\| = \|a\|$ for $a \in A$ positive. Suppose $\|\phi(a)\| < \|a\|$ for some positive $a \in A$. Note that $\phi(a) \geq 0$ since $a = b^*b$ for some $b \in A$, and so

$\phi(a) = \phi(b)^*\phi(b)$. So, the assumption that $\|\phi(a)\| < \|a\|$ is equivalent to the assumption that $r(a) := \alpha > \beta := r(\phi(a))$. Using the continuous functional calculus, we identify $C^*(a) = C_0(\sigma(a)\backslash\{0\}) \subset C_0((0,\alpha])$ and $C^*(\phi(a)) = C_0(\sigma(\phi(a))\backslash\{0\}) \subset C((0,\beta])$. Now, define $f \in C((0,\alpha])$ so that $f|_{(0,\beta]} = 0$, $f(\alpha) = 1$, and f is affine on $[\beta,\alpha]$.

Then

$$\|f(a)\| = \sup_{\lambda\in\sigma(a)} |f(\lambda)| = 1,$$

but

$$\|f(\phi(a))\| = \sup_{\lambda\in\sigma(\phi(a))} |f(\lambda)| = 0.$$

In particular, $f(a) \neq 0$ and $f(a) \in \ker\phi$, contradicting ϕ being injective.

Now, suppose $\pi : A \to B$ is a $*$-homomorphism with kernel $J = \ker(\pi)$. Then A/J is a C*-algebra by Theorem 4.11. Let $q : A \to A/J$ be the quotient map. Then q is a $*$-homomorphism and π factors through the quotient A/J, i.e. there exists a bijective $*$-homomorphism $\rho : A/J \to \pi(A)$ given by $\rho(q(a)) = \pi(a)$. (Indeed, this is just the first isomorphism theorem for algebras. The map ρ is $*$-preserving because q and π are: $\rho(q(a)^*) = \rho(q(a^*)) = \pi(a^*) = \pi(a)^* = \rho(q(a))^*$.)

So, it follows that $\rho : A/J \to B$ is an injective $*$-homomorphism between C*-algebras, which by the first part of this theorem, means that it is isometric. It follows from this that its image $\pi(A)$ is closed in B. □

Exercise 4.14. Why couldn't we get the second statement of Theorem 4.13 as a corollary of Proposition 1.29?

Exercise 4.15. Extend Theorem 4.13 to the general case where the assumptions that A, B, and ϕ are not unital. Here's an idea of what to check. If A is not unital, then we can extend ϕ to $\tilde{A}$ as we did in Proposition 1.22 to map $1 \in \tilde{A}$ to $1 \in B$ or $1 \in \tilde{B}$ depending on whether or not B is unital. If A is unital, then check that $\phi(1)$ is the unit in the C*-subalgebra $C^*(\phi(A)) \subset B$, and we can just replace B with this C*-subalgebra in the proof.

4.1. **Hereditary Subalgebras.** Sometimes, you want to study a C*-algebra that doesn't have any (norm-closed 2-sided) ideals. The next best thing to ideals are hereditary subalgebras. Like the connection between $*$-homomorphisms and cp maps (see Chapter 10), hereditary subalgebras are closely connected to ideals, and can tell you a lot of structural information about your original C*-algebra, but they exist much more often than ideals.

Definition 4.16. A C*-subalgebra B of a C*-algebra A is *hereditary* if whenever $0 \leq a \leq b \in A$ and $b \in B$, then $a \in B$.

Example 4.17. Let A be a unital C*-algebra. For any projection $p \in A$, note first that pAp is a C*-subalgebra of A. In fact, it's a hereditary subalgebra. To see this, choose $0 \leq a \leq b = pcp$. We need to show that $a \in pAp$. By Exercise 3.11, conjugating an inequality in a C*-algebra preserves the inequality, so

$$0 \leq (1-p)a(1-p) \leq (1-p)pcp(1-p) = 0,$$

and therefore $(1-p)a(1-p) = (a^{1/2}(1-p))^*(a^{1/2}(1-p)) = 0$. The C*-identity consequently implies that

$$a^{1/2}(1-p) = 0 = (a^{1/2}(1-p))^* = (1-p)a^{1/2}.$$

We conclude that

$$a^{1/2} = a^{1/2}p = pa^{1/2}.$$

In other words, $a = (a^{1/2})^2 = pap \in pAp$ whenever $a \leq pcp$ for some $c \in A$.

The following theorem gives us the promised connection between hereditary subalgebras and ideals. We won't prove it in lecture, but you can find a very friendly proof in [11, Theorem 3.2.1]. (In fact, this section is based on [11, Section 3.2], with the only major change being to include a few more details.)

Theorem 4.18. *If L is a norm-closed left ideal in A, then $L \cap L^*$ is a hereditary subalgebra of A. Moreover, if $B \leq A$ is a hereditary subalgebra, then*

$$\{a \in A : a^*a \in B\}$$

is a closed left ideal of A. This gives a bijective correspondence between hereditary subalgebras and closed left ideals, which respects inclusions.

Theorem 4.19. *A* C^**-subalgebra* B *of* A *is hereditary iff* $bab' \in B$ *for all* $b, b' \in B$ *and* $a \in A$.

Proof. Suppose $B \leq A$ is hereditary. Then Theorem 4.18 tells us $B = L \cap L^*$ for some left ideal L. So if $b, b' \in B$ we have $b(ab') \in L$ and $(bab')^* = (b')^* a^* b^* \in L$ [since $b^* \in B \subseteq L$], and therefore $bab' \in L \cap L^* = B$.

On the other hand, if $bab' \in B$ for all $b, b' \in B$ and $a \in A$, choose $0 \leq a \leq b$ with $b \in B$. Let $(e_\lambda)_\lambda$ be an approximate unit for B. Again, applying Exercise 3.11, we see that

$$0 \leq (1 - e_\lambda)a(1 - e_\lambda) \leq (1 - e_\lambda)b(1 - e_\lambda)$$

and (since $(1 - e_\lambda)a(1 - e_\lambda) = (a^{1/2}(1 - e_\lambda)^*(a^{1/2}(1 - e_\lambda))$

$$\|a^{1/2} - a^{1/2}e_\lambda\| \leq \|b^{1/2} - b^{1/2}e_\lambda\|$$

for all λ. However, since $b^{1/2} \in B$, $\lim_\lambda b^{1/2}e_\lambda = b^{1/2}$. It follows that $a^{1/2} = \lim_\lambda a^{1/2}e_\lambda$, or in other words $a = \lim_\lambda e_\lambda a e_\lambda$. By our hypothesis, $e_\lambda a e_\lambda \in B$ for all λ, so the fact that B is a norm-closed $*$-subalgebra means that $a \in B$, as desired. □

An immediate corollary is the following:

Corollary 4.20. *If* I *is a norm-closed 2-sided ideal of a* C^**-algebra* A*, then* I *is a hereditary subalgebra.*

Corollary 4.21. *If* $b \in A_+$*, then* $\overline{bAb}$ *is a hereditary subalgebra. In fact it's the smallest hereditary subalgebra of* A *containing* b.

Proof. Use an approximate unit to show that $b^2 \in \overline{bAb}$; the functional calculus then tells you that $b \in \overline{bAb}$. Now apply Theorem 4.19 to see that $\overline{bAb}$ is hereditary, and that if B is any other hereditary subalgebra of A with $b \in B$, we must have $\overline{bAb} \in B$. □

In fact, in the separable case, every hereditary subalgebra is of this form.

Theorem 4.22. *If* B *is a separable hereditary subalgebra of* A*, then there is* $b \in B_+$ *such that* $B = \overline{bAb}$.

Proof. Choose a sequential approximate unit $(e_n)_{n\in\mathbb{N}}$ for B. Define $b = \sum_n e_n 2^{-n} = \lim_{N\to\infty} \sum_{n=1}^N e_n 2^{-n}$. The fact that B_+ is a closed cone implies that $b \in B_+$, so Corollary 4.21 implies that $\overline{bAb} \subseteq B$.

To see equality, notice first that $e_n 2^{-n} \leq b$ for all $n \in \mathbb{N}$, and so the fact that $\overline{bAb}$ is hereditary means that $e_n \in \overline{bAb}$. Theorem 4.19 then implies that for any $d \in B$ we have $e_n d e_n \in \overline{bAb}$, and since $\overline{bAb}$ is norm-closed, we conclude that

$$\forall d \in B,\ d = \lim_n e_n d e_n \in \overline{bAb}.$$ □

Exercise 4.23. Suppose A is a separable C^*-algebra and $p \in A$ is a projection. Show that pAp is closed.

Remark 4.24. We can now give another definition of σ-unital. It turns out (requires proof) that $b \in A$ is strictly positive iff $\overline{bAb} = A$. Since any ideal is a hereditary subalgebra, we can see how strictly positive elements generalize positive invertible elements.

4.2. Representations.

Definition 4.25. A *representation* of a C^*-algebra A is a $*$-homomorphism $\pi : A \to B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. A representation π is

- *nondegenerate* if $\pi(A)\mathcal{H}$ is dense in $\mathcal{H}$, or equivalently if for any $\xi \in \mathcal{H}$, if $\pi(a)\xi = 0$, for all $a \in A$ then $\xi = 0$
- *(topologically) irreducible* if it has no closed invariant subspace
- *faithful* if it is injective (and hence an isometric embedding).

If A is unital, we say π is *unital* when $\pi(1_A) = I \in B(\mathcal{H})$.

We will discuss irreducible representations more in Chapter 8. A paradigm example of a degenerate representation is where $\mathcal{H}$ decomposes as a nontrivial direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2$ and $\pi(A)$ can be realized as a $*$-subalgebra of operators on $B(\mathcal{H}_1)$ identified with the operators whose kernels contain $\mathcal{H}_2$.

Exercise 4.26. Show that any unital representation is nondegenerate. Show that any irreducible representation is nondegenerate.

Remark 4.27. Non-degeneracy is a common assumption, which avoids some obnoxious pitfalls. Many times theorems which are phrased for nondegenerate representations still hold without this assumption. The trick usually amounts to taking a degenerate representation $\pi : A \to B(\mathcal{H})$ and to define its restriction to the closure of $\pi(A)\mathcal{H}$. Some delicacy may be required after this, depending on what statement you are trying to prove. We will point out an example later. (Theorem 12.6.)

Remark 4.28. A representation is topologically irreducible iff $\pi(A)' = \mathbb{C}$ (c.f. [8, Lemma 1.9.1]) iff $\pi(A)$ has no invariant subspaces (although the proof of this statement needs Kadison Transitivity).

Exercise 4.29. A family of representations $\{\pi_i : A \to B(\mathcal{H}_i)\}_{i\in I}$ for a C*-algebra A is *separating* if for any $a, b \in A$, there exists $i \in I$ such that $\pi_i(a) \neq \pi_i(b)$. Define $\pi : A \to B(\oplus_i \mathcal{H}_i)$ by $\pi(a) = \oplus_i \pi_i(a)$. Show that π is a *faithful* representation, i.e. an isometric representation, if the family $\{\pi_i\}_{i\in I}$ is separating.

Now, suppose $\{a_j\}_{j\in J}$ is a dense subset of A. We cannot conclude from knowing that $\{\pi_i\}_{i\in I}$ is separating for $\{a_j\}_{j\in J}$ that π is faithful (why?). However, if we know that for each $j \in J$, there exists $i \in I$ such that $\|\pi_i(a_j)\| = \|a_j\|$, then we can conclude that π is faithful (why?).

5. Group C*-algebras

Preview of Lecture:

In lecture, we'll discuss Proposition 5.10 and Example 5.14.

Most of the steps of the proof of Proposition 5.25 are relatively straightforward; the one which requires the most creativity is the fact that $h(\omega) \in \widehat{C_r^*(G)}$ for all $\omega \in \widehat{G}$ so we'll discuss that in lecture.

In this and later sections, we will want to consider ℓ^p-spaces of discrete topological groups.[7] Here is a quick refresher for how to think of them.

Let G be a discrete group. For each $g \in G$, we can define a function $\delta_g : G \to \mathbb{C}$ by

$$\delta_g(x) = \begin{cases} 1 & \text{if } x = g \\ 0 & \text{if else} \end{cases}$$

For $1 \le p < \infty$, we define

$$\begin{aligned} \ell^p(G) &:= \{f : G \to \mathbb{C} \mid \sum_{g\in G} |f(g)|^p < \infty\} \\ &= \{(a_g)_{g\in G} \mid a_g \in \mathbb{C} \ \forall\, g \in G, \ \text{ and } \ \sum_{g\in G} |a_g|^p < \infty\} \\ &= \{\sum_{g\in G} a_g\delta_g \mid a_g \in \mathbb{C} \ \forall\, g \in G, \ \text{ and } \ \sum_{g\in G} |a_g|^p < \infty\} \\ &= \overline{\text{span}}^{\|\cdot\|_p}\{\delta_g \mid g \in G\}, \end{aligned}$$

and for $p = \infty$, we have

$$\begin{aligned} \ell^\infty(G) &:= \{f : G \to \mathbb{C} \mid \sup_{g\in G} |f(g)| < \infty\} \\ &= \{(a_g)_{g\in G} \mid a_g \in \mathbb{C} \ \forall\, g \in G, \ \text{ and } \ \sup_{g\in G} |a_g| < \infty\} \\ &= \{\sum_{g\in G} a_g\delta_g \mid a_g \in \mathbb{C} \ \forall\, g \in G, \ \text{ and } \ \sup_{g\in G} |a_g| < \infty\}. \end{aligned}$$

5.1. C*-algebras associated to discrete groups.

A useful source of examples and motivation for C*-theory are the *group* C*-*algebras*, i.e., C*-algebras arising from topological groups. Indeed, one can view a group C*-algebra as encoding the (infinite-dimensional) representations of the group. (See Exercise 5.17.) Understanding these representations better was a main motivation for a lot of the early work on C*-algebras, and group C*-algebras are still a fundamental source of examples and inspiration for research today.

Definition 5.1. Let G be a discrete group. The *complex group algebra* $\mathbb{C}G$ is the algebra generated by $\{u_g : g \in G\}$, where $u_g u_h = u_{gh}$.

By definition, then, $\mathbb{C}G$ consists of all finite products of finite linear combinations of $\{u_g : g \in G\}$, i.e., sums of the form $\sum_{g\in G} a_g u_g$ where the $a_g \in \mathbb{C}$ are zero for all but finitely many terms.

Remark 5.2. Note that G embeds naturally into $\mathbb{C}G$ by $g \mapsto u_g$. In fact, one would normally say $\mathbb{C}G$ is the algebra consisting of formal linear combinations $\sum_{g\in G} a_g g$ of elements of G (with only finitely many non-zero a_g). We introduce this u_g notation here because there will soon be a few copies of G floating around, and we will want to tell them apart.

Observe that $\mathbb{C}G$ is always unital (what's the unit?). Moreover, we have a natural involution on $\mathbb{C}G$:

$$(a_g u_g)^* := \overline{a_g} u_{g^{-1}}$$

where $a_g \in \mathbb{C}$. (Check for yourself that this formula indeed gives an involution.)

[7] A topological group is just a group equipped with a topology so that the group operations (multiplication and inverses) are continuous with respect to this topology. In order to do analysis on these spaces, we require that the group is locally compact and Hausdorff. A favorite class of examples are the discrete groups, i.e., ones equipped with the discrete topology.

Given $\sum_{g\in G} a_g u_g, \sum_{h\in G} b_h u_h \in \mathbb{C}G$, the formula for the multiplication of the generators $\{u_g\}_{g\in G}$ implies that

$$\left(\sum_{g\in G} a_g u_g\right)\left(\sum_{h\in G} b_h u_h\right) = \sum_{g\in G}\sum_{h\in G} a_g b_h u_g u_h = \sum_{g\in G}\sum_{h\in G} a_g b_h u_{gh} \tag{5.1}$$

$$(h \leftrightarrow g^{-1}h) \qquad = \sum_{g\in G}\sum_{h\in G} a_g b_{g^{-1}h} u_h \tag{5.2}$$

$$\text{finite sums} \qquad = \sum_{h\in G}\left(\sum_{g\in G} a_g b_{g^{-1}h}\right) u_h. \tag{5.3}$$

This multiplication may look familiar if you've seen convolution multiplication or the Fourier transform before. For functions ϕ, ψ on a discrete group G, their *convolution product* is

$$\phi * \psi(g) := \sum_{h\in G} \phi(h)\psi(h^{-1}g).$$

If we think of the coefficients $(a_g)_{g\in G}$ of an element $\phi = \sum_{g\in G} a_h u_h \in \mathbb{C}G$ as a function from G to $\mathbb{C}$, then the function associated to the product $(\sum_{h\in G} a_g u_g)(\sum_{g\in G} b_g u_g)$ is precisely the convolution product of the functions $\phi = (a_g)_{g\in G}$ and $\psi = (b_g)_{g\in G}$.

If we want to complete the $*$-algebra $\mathbb{C}G$ into a C*-algebra, we first need a norm, which we get from a representation.

Definition 5.3. A *representation* of a $*$-algebra A is a $*$-preserving homomorphism $\pi : A \to B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. If A is unital, we will assume π is *unital* in that it takes the unit of A to the unit of $B(\mathcal{H})$. If π is injective we say that it is *faithful*.

Note that if π is a representation of $\mathbb{C}G$ and $a \in \mathbb{C}G$, then the fact that $B(\mathcal{H})$ is a C*-algebra implies that

$$\|\pi(a^*a)\| = \|\pi(a)^*\pi(a)\| = \|\pi(a)\|^2.$$

In particular, the norm on A induced by π, $\|a\|_\pi := \|\pi(a)\|$, satisfies the C*-identity. Therefore,

$$C^*_\pi(G) := \overline{\pi(\mathbb{C}G)}$$

is a C*-algebra.

Exercise 5.4. If π is a representation of $\mathbb{C}G$, what sort of operator will $\pi(u_g)$ be? Can you say anything about $\|\pi(u_g)\|$?

There is a natural representation of $\mathbb{C}G$ on $\ell^2(G)$ called the *left regular representation* and often denoted by λ: On the generators, we define

$$\lambda(u_g)(\delta_h) = \delta_{gh},$$

and extend λ to $\mathbb{C}G$ by requiring it to also be linear.

Remark 5.5. This should remind you of the action induced on $\ell^\infty(G)$ by the action of G on itself by left-multiplication, which we saw in Exercise 7.11 from the Prerequisite notes. This is why it's called the "left regular" representation. Yes, there is also a right regular representation, which is occasionally of interest.

Exercise 5.6. What is the adjoint of $\lambda(u_g)$? Is λ $*$-preserving?

Observe (check!) that λ is injective. So, we can think of $\mathbb{C}G$ as a subalgebra of $B(\ell^2(G))$. The *reduced group* C*-*algebra* $\mathrm{C}^*_r(G)$ is defined to be

$$\mathrm{C}^*_r(G) := \overline{\lambda(\mathbb{C}G)}.$$

So that we don't always have to choose a specific representation (and for abstract-nonsense reasons) we often want to work with the *universal (or maximal) group* C*-*algebra* $\mathrm{C}^*(G)$, (sometimes written $\mathrm{C}^*_u(G)$ or $\mathrm{C}^*_{\max}(G)$) which is defined to be the completion of $\mathbb{C}G$ in the *universal norm*

$$\|a\|_u := \sup\{\|\pi(a)\| : \pi \text{ a representation of } \mathbb{C}G\}. \tag{5.4}$$

Since we know any discrete group admits *some* representation (i.e., the left-regular one), the supremum is over a non-empty set. However, a reader who is familiar with set theory might notice that we have made no assertion about whether the collection of all representations of $\mathbb{C}G$ is a set. How, then, do we know that we can take the supremum in (5.4)? Recall that, for any $a \in \mathbb{C}G$ and any representation π of $\mathbb{C}G$, the quantity $\|\pi(a)\|$ is a real number, being the norm of an operator on some Hilbert space. So the collection in (5.4) is a subclass of the set of all real numbers, and basic results from set theory guarantee that a subclass of a set is still a set. It follows that the universal norm is well defined.

OK, fine, but how do we know that the universal norm is finite? In fact, the universal norm is bounded above by the ℓ^1 norm:

Proposition 5.7. *If π is a representation of $\mathbb{C}G$, then for any finite set $F \subseteq G$ and any $a = \sum_{g\in F} a_g u_g \in \mathbb{C}G$ we have $\|\pi(a)\| \leq \sum_{g\in F} |a_g|$.*

Proof. Since $\pi(u_g)$ is a unitary for all g, and hence has norm 1, the triangle inequality tells us that

$$\|\pi(a)\| \leq \sum_{g\in F} \|a_g u_g\| = \sum_{g\in F} |a_g|.$$

□

It follows that if a net in $\mathbb{C}G$ is Cauchy in the ℓ^1 norm, then that net is also Cauchy in $\mathrm{C}^*(G)$ (and $\mathrm{C}^*_r(G)$). In other words, we could alternatively think of $\mathrm{C}^*(G)$ and $\mathrm{C}^*_r(G)$ as completions in a C*-norm of $\ell^1(G)$. This will come in handy sometimes, for example in Section 5.2.

Proposition 5.8. *The "max" norm is a C*-norm.*

Proof. Let $\{(\pi_\lambda, H_\lambda) : \lambda \in \Lambda\}$ be the collection of $*$-representations of $\mathbb{C}G$, and define $H_u := \oplus_\lambda H_\lambda$. Recall that $||(\xi_\lambda)||_u^2 = \sum_\lambda ||\xi_\lambda||^2$. Define $\pi_u : \mathbb{C}G \to B(H_u)$ by $\pi_u(f)(\xi_\lambda) := (\pi_\lambda(f)\xi_\lambda)$, and note that $\pi_u(f)$ is linear and bounded with $||\pi_u(f)|| = \sup\{||\pi_\lambda(f)|| : \lambda \in \Lambda\}$. Furthermore, π_u is a $*$-representation by virtue of the π_λ all being $*$-homomorphisms. Therefore, $||a||_u = ||\pi_u(a)||$ is a C*-norm. □

Proposition 5.9. *$\mathbb{C}G$ is dense in both $\mathrm{C}^*_r(G)$ and $\mathrm{C}^*(G)$.*

The reason we call $\mathrm{C}^*(G)$ the "universal group C*-algebra" is the following proposition. While the argument used in the proof is straightforward, it's a very powerful technique for constructing $*$-homomorphisms out of many examples of C*-algebras, not just group C*-algebras.

Proposition 5.10. *For any representation π of $\mathbb{C}G$, there is an associated surjective $*$-homomorphism $\hat{\pi} : \mathrm{C}^*(G) \to \mathrm{C}^*_\pi(G)$.*

Proof. We define $\hat{\pi}$ first for $a \in \mathbb{C}G \subseteq \mathrm{C}^*(G)$:

$$\hat{\pi}(a) := \pi(a) \in \mathrm{C}^*_\pi(G).$$

As π is a representation of $\mathbb{C}G$, in order to extend $\hat{\pi}$ to a $*$-homomorphism on all of $\mathrm{C}^*(G)$, I claim that it suffices to check that $\hat{\pi}$ is norm-decreasing on $\mathbb{C}G \subseteq \mathrm{C}^*(G)$. Why? Well, once we know that $\|\hat{\pi}(a)\| \leq \|a\|_u$ for all $a \in \mathbb{C}G$, then if $x \in \mathrm{C}^*(G)$ is a norm limit of elements in $\mathbb{C}G$, $x = \lim_i a_i$, then in particular, given any $\varepsilon > 0$, we can find I such that $\|a_i - a_j\|_u < \varepsilon$ whenever $i, j \geq I$. If $\hat{\pi}$ is norm-decreasing on $\mathbb{C}G \subseteq \mathrm{C}^*(G)$, then it follows that $(\hat{\pi}(a_i))_i$ is Cauchy in $\mathrm{C}^*_\pi(G)$. As $\mathrm{C}^*_\pi(G)$ is complete, $\lim_i(\hat{\pi}(a_i))_i$ has a limit, call it y. Defining $\hat{\pi}(x) := y$, one can check that $\hat{\pi}(x)$ is independent of the approximating Cauchy sequence $(a_i)_i \subseteq \mathbb{C}G \subseteq \mathrm{C}^*(G)$, and that this definition makes $\hat{\pi}$ into a $*$-homomorphism.

Now the definition of the universal norm tells us immediately that

$$\|\hat{\pi}(a)\| = \|\pi(a)\| \leq \|a\|_u, \ \forall\, a \in \mathbb{C}G.$$

□

Exercise 5.11. Fill in the gaps in the proof of Proposition 5.10. (This includes checking that $\hat{\pi}$ is surjective.)

Exercise 5.12. Let G be a discrete group. Then $\widehat{\mathrm{C}^*(G)} \neq \emptyset$ (whether or not G is abelian).

Remark 5.13. In general, if π_1 and π_2 are two unitary representations of $\mathbb{C}G$, then the identity map $\mathbb{C}G \to \mathbb{C}G$ extends to a surjection $\mathrm{C}^*_{\pi_1}(G) \to \mathrm{C}^*_{\pi_2}(G)$ iff $\|\pi_1(a)\| \geq \|\pi_2(a)\|$ for all $a \in \mathbb{C}G$.

Example 5.14. Let $G = \mathbb{Z}$ (under addition). Observe that if $u \in B(\mathcal{H})$ is a unitary, then we obtain a representation $\pi : \mathbb{C}\mathbb{Z} \to B(\mathcal{H})$ given by defining $\pi(u_1) = u$ (where u_1 corresponds to the cyclic generator of $\mathbb{Z}$). Conversely, any representation π of $\mathbb{C}\mathbb{Z}$ arises in this way.

It follows that, for any unitary u, there is a surjective $*$-homomorphism $\hat{\pi} : \mathrm{C}^*(\mathbb{Z}) \to \mathrm{C}^*(\{u\})$. In other words, $\mathrm{C}^*(\mathbb{Z})$ is the *universal* C*-*algebra generated by a unitary.*

Now, consider $C^*_r(\mathbb{Z})$. The Fourier transform $\mathcal{F}$ gives us a unitary isomorphism $\mathcal{F} : \ell^2(\mathbb{Z}) \to L^2(\mathbb{T})$,

$$\mathcal{F}(\xi)(z) = \sum_{n\in\mathbb{Z}} \xi_n z^n, \quad \text{for} \quad \xi = (\xi_n)_{n\in\mathbb{Z}}, \ z \in \mathbb{T},$$

which takes convolution multiplication to pointwise multiplication. Then the Fourier transform implements an isomorphism $B(\ell^2(\mathbb{Z})) \to B(L^2(\mathbb{T}))$. Where does this isomorphism send $C^*_r(\mathbb{Z}) \subset B(\ell^2(\mathbb{Z}))$?

As we saw in Example 1.37 from the Prerequisite notes, for $f \in C(\mathbb{T})$, we can define the operator $M_f \in B(L^2(\mathbb{T}))$ by

$$M_f\eta(z) = f(z)\eta(z), \quad \text{for } \eta \in L^2(\mathbb{T}), \ z \in \mathbb{T}.$$

Because it takes convolution multiplication to pointwise multiplication, the Fourier transform implements an isomorphism

$$C^*_r(\mathbb{Z}) \cong \{M_f : f \in C(\mathbb{T})\} \subseteq B(L^2(\mathbb{T})).$$

Even without the Fourier transform, one easily checks that the $*$-algebra structure on $\{M_f : f \in C(\mathbb{T})\}$ agrees with the $*$-algebra structure on $C(\mathbb{T})$, and $\|M_f\| = \|f\|_\infty$, so $\{M_f : f \in C(\mathbb{T})\} \cong C(\mathbb{T})$ as C*-algebras.

Finally, consider the C*-algebra $C(\mathbb{T})$. The Stone-Weierstrass Theorem (cf. [6, Theorem I.5.6]) tells us that $C(\mathbb{T})$ is generated, as a C*-algebra, by the function

$$f(z) = z.$$

It turns out (see Section 5.2) that $C(\mathbb{T})$ can also be described as the universal C*-algebra generated by a unitary. That is,

$$C^*(\mathbb{Z}) \cong C^*_r(\mathbb{Z}) \cong C(\mathbb{T}).$$

Proposition 5.15. *If $G \leq H$ then $C^*(G)$ is a norm-closed subalgebra of $C^*(H)$. The same is true for the reduced C*-algebras.*

Proof. Let $\iota : \mathbb{C}G \to \mathbb{C}H$ denote the canonical inclusion. We first claim that if we view $\mathbb{C}G$ (respectively $\mathbb{C}H$) as a subalgebra of $C^*(G)$ (resp. $C^*(H)$), then ι is norm-decreasing. It then follows (using the same argument as in Proposition 5.10) that ι induces an $*$-homomorphism $\tilde{\iota} : C^*(G) \to C^*(H)$.

To see that ι is norm-decreasing, observe that every representation of $\mathbb{C}H$ restricts to a representation of $\mathbb{C}G$. Thus, the set used in (5.4) to compute the universal norm for G contains the set

$$\{\|\pi(a)\| : \pi \text{ a representation of } \mathbb{C}G \text{ which extends to a representation of } \mathbb{C}H\}.$$

It follows that $\|\iota(a)\|_{u,H} \leq \|a\|_{u,G}$ for all $a \in \mathbb{C}G$.

The proof that $\tilde{\iota}$ is injective will be relatively straightforward once we've proved the Gelfand-Naimark-Segal Theorem, so we'll come back to it. □

Here are two more structural results about $C^*(G)$.

Proposition 5.16.

(1) $C^*(G)$ is never simple unless $G = \{e\}$ is trivial.
(2) If $|G| = n$, then $C^*(G) \cong C^*_r(G)$ embeds as a C-subalgebra of M_n.*
(3) If $|G| = n$ and is abelian, then $C^*(G) \cong \mathbb{C}^n$.

Proof. (1) For any group G, there is a representation π of $\mathbb{C}G$ on $\mathbb{C}$, given by

$$\pi(u_g) = 1, \qquad \forall\, g \in G.$$

Observe that π is onto. If $G \neq \{e\}$, then we can choose $g \neq h \in G$, and

$$u_g - u_h \in \ker \pi.$$

Thus, $\ker \pi$ is a nontrivial ideal in $C^*(G)$.

(2) If $|G| = n$, then $\ell^2(G) \cong \mathbb{C}^n$, and so $\lambda : \mathbb{C}G \to B(\ell^2(G)) \cong M_n(\mathbb{C})$ is a faithful embedding of $\mathbb{C}G$ into $M_n(\mathbb{C})$. Since $M_n(\mathbb{C})$ is finite dimensional, $\mathbb{C}G$ is already complete, and hence is a C*-algebra. By Remark 1.32, this means that $C^*_r(G) = \mathbb{C}G = C^*(G)$.

(3) Suppose $|G| = n$ and G is also abelian. There are a couple of ways we could prove that $G \cong \mathbb{C}^n$, one using C*-techniques, and one using linear algebra.

Option 1: It follows (Exercise 5.19) that $\mathbb{C}G$ is also abelian and hence so is $C^*(G)$. Since every finite dimensional C*-algebra is a direct sum of matrix algebras by Proposition 8.5 and any nontrivial matrix algebra is nonabelian, the result follows.

Option 2: First, consider the case where $G = \mathbb{Z}_n$. Since G is cyclic, $\mathbb{C}G \subset \mathrm{M}_n(\mathbb{C})$ is generated as an algebra by a single unitary, u_1 (the permutation matrix that sends the basis elements $e_i \mapsto e_{(i+1) \bmod k}$). Since u_1 is a unitary, it is unitarily diagonalizable, i.e., there exists a unitary $v \in \mathrm{M}_n(\mathbb{C})$ so that $vu_1v^* = \mathrm{diag}(\omega_1, ..., \omega_n)$ where $\omega_1, ..., \omega_n$ are the eigenvalues of u_1 (in this case the n^{th} roots of unity). It follows that $\mathrm{C}^*(\mathbb{Z}_n)$ is isomorphic to the subalgebra of $M_n(\mathbb{C})$ generated by $\mathrm{diag}(\omega_1, ..., \omega_n)$, which is exactly $\mathbb{C}^n$.

Now, if G is abelian and $|G| = n$, then the fundamental theorem of finite abelian groups tells us that $G \cong \mathbb{Z}_{k_1} \times \ldots \times \mathbb{Z}_{k_m}$ for some $n \in \mathbb{N}$ and $k_1, ..., k_m$ with $\sum_{j=1}^m k_j = n$. Then $\pi := \oplus_{j=1}^m \lambda_{\mathbb{Z}_{k_j}} : \mathbb{Z}_{k_1} \times \ldots \times \mathbb{Z}_{k_m} \to \oplus_{j=1}^m M_{k_j}(\mathbb{C})$ is a representation of G, which means we can identify $\mathrm{C}^*(G)$ with $\mathrm{C}^*_\pi(G) = \oplus_{j=1}^m \mathrm{C}^*_r(\mathbb{Z}_{k_j}) \cong \oplus_{j=1}^m \mathbb{C}^{k_j} = \mathbb{C}^n$.

□

Exercise 5.17. Recall that the set $U(\mathcal{H})$ of unitaries in $B(\mathcal{H})$ is a group under multiplication. A *unitary representation* of a group G is a group homomorphism $\rho : G \to U(\mathcal{H})$. Show that representations of $\mathbb{C}G$ are in bijection with unitary representations of G.

Remark 5.18. In this section we've focused on discrete groups and their C*-algebras. However, one can also define the group C*-algebra for any group G which has a locally compact Hausdorff topology with respect to which multiplication and inversion are continuous (for short, these are called *locally compact* groups). While a lot of the theory of (discrete) group C*-algebras goes through smoothly in the locally compact setting, Proposition 5.15 is a major exception: it is not true for locally compact groups. For example, consider $\mathbb{R}$ under addition. It turns out that $\mathrm{C}^*(\mathbb{R}) = C_0(\mathbb{R})$, and $\mathbb{Z}$ is a subgroup of $\mathbb{R}$, but $\mathrm{C}^*(\mathbb{Z}) \cong C(\mathbb{T})$ is not a subalgebra of $C_0(\mathbb{R})$. This example highlights the other major exception: Proposition **??**. Notice that $C_0(\mathbb{R})$ is not unital. In particular, it contains no units, let alone a copy of $\mathbb{R}$– that's right, $\mathrm{C}^*(\mathbb{R})$ does not contain $\mathbb{R}$.

5.2. **Abelian group C*-algebras.** If G is abelian, then $u_g u_h = u_h u_g$ for all $g, h \in G$, and so $\mathbb{C}G$ is also abelian.

Exercise 5.19. Show that any C*-completion of $\mathbb{C}G$ is an abelian C*-algebra.

By Exercise 5.19 and the Gelfand-Naimark Theorem (Theorem 2.11), it follows that $\mathrm{C}^*_r(G) = C_0(\widehat{\mathrm{C}^*_r(G)})$ and $\mathrm{C}^*(G) = C_0(\widehat{\mathrm{C}^*(G)})$ for some locally compact Hausdorff spaces $\widehat{\mathrm{C}^*_r(G)}$ and $\widehat{\mathrm{C}^*(G)}$, respectively. In fact, these spaces must be compact since $\mathrm{C}^*(G)$ and $\mathrm{C}^*_r(G)$ are unital. But what are the spaces $\widehat{\mathrm{C}^*_r(G)}$ and $\widehat{\mathrm{C}^*(G)}$ exactly, and do they have anything to do with G?

Definition 5.20. For an abelian group G, $\widehat{G}$ denotes the *Pontryagin dual* of G:

$$\widehat{G} = \{\omega : G \to \mathbb{T} \text{ group homomorphism}\}. \tag{5.5}$$

Exercise 5.21. Show that $\widehat{G}$ is also a group, under pointwise multiplication. Do you need to assume G is abelian?

Our next main goal is to prove that $\widehat{G}$ and $\widehat{\mathrm{C}^*(G)}$ are homeomorphic (Proposition 5.22). Then we will prove (Proposition 5.25) that $\widehat{G}$ and $\widehat{\mathrm{C}^*_r(G)}$ are also homeomorphic. (Notice that this will prove that $\mathrm{C}^*(G) \cong \mathrm{C}^*_r(G)$ for any abelian group G! In fancy language, we are proving that abelian groups are *amenable* – see Chapter 13.) In order to do that, we need to identify the topology on $\widehat{G}$.

The topology on $\widehat{G}$ (when G is discrete) is the *point-norm topology*: a net $(\omega_i)_{i\in\Lambda} \subseteq \widehat{G}$ is Cauchy iff, for all $g \in G$, the nets $(\omega_i(g))_{i\in\Lambda} \subseteq \mathbb{T}$ are Cauchy.[8] Equivalently, a basis for the topology on $\widehat{G}$ consists of the sets

$$B_{\varepsilon,F}(\omega) := \{\eta \in \widehat{G} : |\eta(g) - \omega(g)| < \varepsilon \ \forall\ g \in F \text{ finite}\}.$$

Note that this topology is Hausdorff. (Indeed, if $\eta \neq \omega$, then they differ at some $g \in G$. Set $\varepsilon < |\eta(g) - \omega(g)|$, and $B_{\varepsilon,\{g\}}(\eta) \cap B_{\varepsilon,\{g\}}(\omega) = \emptyset$.)

[8] If G is abelian but not discrete, its Pontryagin dual still exists, but the topology is that of uniform convergence on compact sets. For discrete groups, these are the same.

Proposition 5.22. *The map* $\Omega : \widehat{\mathrm{C}^*(G)} \to \widehat{G}$ *given by*

$$\Omega(\phi)(g) = \phi(u_g), \ \forall \ \phi \in \widehat{\mathrm{C}^*(G)}, g \in G$$

is a homeomorphism.

Remark 5.23. Note that the map $\pi : G \to \{u_g \mid g \in G\}$ given by $g \mapsto u_g$ is a group isomorphism, and so Ω is essentially the map $\phi \mapsto \phi \circ \pi$. Believably enough, the Pontryagin duals of two isomorphic groups are homeomorphic. So, if we replace G with $\{u_g \mid g \in G\} \subset \mathrm{C}^*(G)$, we can re-interpret the map Ω above as just the restriction of characters on $\mathrm{C}^*(G)$ to this copy of G in $\mathrm{C}^*(G)$.

Proof. Since $\widehat{\mathrm{C}^*(G)}$ is compact and $\widehat{G}$ is Hausdorff, it suffices to show that Ω is a continuous bijection. Notice that any $\phi \in \widehat{\mathrm{C}^*(G)}$ is a representation $\phi : \mathrm{C}^*(G) \to B(\mathbb{C}) = \mathbb{C}$. Since ϕ is a homomorphism (Corollary 2.25), it follows that $\Omega(\phi)$ is a homomorphism. Moreover, since each u_g is a unitary in $\mathrm{C}^*(G)$, (c.f. Exercise 5.17), it follows that $\phi(u_g) \in \mathcal{U}(B(\mathbb{C})) = \mathbb{T}$ for all $g \in G$. So, Ω is well-defined. If $\phi, \psi \in \widehat{\mathrm{C}^*(G)}$ and $\Omega(\phi) = \Omega(\psi)$, then $\phi(u_g) = \psi(u_g)$ for all $g \in G$. Linearity implies that $\phi|_{\mathbb{C}G} = \psi|_{\mathbb{C}G}$, and then density and continuity tell us $\phi = \psi$. So, Ω is injective.

To see that it is surjective, note that any $\omega \in \widehat{G}$ is a unitary representation of G (again since $\mathcal{U}(B(\mathbb{C})) = \mathbb{T}$). From Exercise 5.4, this induces a unitary representation $\phi_\omega : \mathbb{C}G \to B(\mathbb{C}) = \mathbb{C}$ given by extending the map $u_g \mapsto \omega(g)$ linearly.[9] Proposition 5.10 says this extends to a *-homomorphism $\phi : \mathrm{C}^*(G) \to \mathbb{C}$ (meaning $\phi(a) = \phi_\omega(a)$ for all $a \in \mathbb{C}G$). In particular, $\phi(u_g) = \omega(g)$ for all $g \in G$. Hence the map is surjective.

It remains to show that Ω is continuous. Suppose that $(\phi_i)_i \subseteq \widehat{\mathrm{C}^*(G)}$ is Cauchy in the weak*-topology–that is, for any $a \in \mathrm{C}^*(G)$ the net $(\phi_i(a))_i \subseteq \mathbb{C}$ is Cauchy. In particular, the net

$$(\phi_i(u_g))_i = (\Omega(\phi_i)(g))_i \subseteq \mathbb{T}$$

is Cauchy for each $g \in G$. Hence Ω is continuous. □

Exercise 5.24. Where does the above proof break down if you use $\mathrm{C}^*_r(G)$ instead of $\mathrm{C}^*(G)$?

Proposition 5.25. *The map* $h : \widehat{G} \to \widehat{\mathrm{C}^*_r(G)}$ *given by, for* $\omega \in \widehat{G}$ *and* $a = \sum_{g\in F} a_g u_g \in \mathbb{C}G$,

$$h(\omega)(a) = \sum_{g\in G} a_g \omega(g), \tag{5.6}$$

is a homeomorphism of topological spaces.

Proof. We first need to show that the formula for $h(\omega)$ given in Equation (5.6) does indeed define an element of $\widehat{\mathrm{C}^*_r(G)}$. We begin by showing that $h(\omega)$ is a $*$-algebra homomorphism. If $b = \sum_{g\in G} b_g u_g$ is another element of $\mathbb{C}G$,

$$h(\omega)(ab) = \sum_{g\in G}\left(\sum_{h\in G} a_h b_{h^{-1}g}\right)\omega(g),$$

whereas the fact that ω is a group homomorphism implies that

$$h(\omega)(a)\cdot h(\omega)(b) = \left(\sum_{g\in G} a_g\omega(g)\right)\left(\sum_{h\in G} b_h\omega(h)\right) = \sum_{k\in G}\left(\sum_{h\in G} a_{kh^{-1}} b_h\right)\omega(k).$$

Making the change of variable $h \mapsto h^{-1}k$, we see that $h(\omega)(ab) = h(\omega)(a)\cdot h(\omega)(b)$ as claimed. Similarly, since $\omega(g^{-1}) = \omega(g)^{-1} = \overline{\omega(g)}$,

$$h(\omega)(a^*) = \sum_{g\in G}\overline{a_g}\omega(g^{-1}) = \overline{\sum_{g\in G} a_g\omega(g)} = (h(\omega)(a))^*.$$

To see that our formula for $h(\omega)$ extends to a bounded linear functional on $\mathrm{C}^*_r(G)$, we need to show that $|h(\omega)a| \leq \|a\|_r$ for all $a \in \mathbb{C}G$. To that end, we first observe that for any $\chi \in \widehat{\mathrm{C}^*_r(G)}$, if we define

$$\tilde{a} = \sum_{g\in G} a_g\omega(g)\overline{\chi(u_g)}u_g,$$

[9]This what we implicitly did when we defined the left regular representation.

then $h(\omega)(a) = \chi(\tilde{a})$. Since the Gelfand transform is isometric, it follows that

$$\|\tilde{a}\|_r = \sup\{|\eta(\tilde{a})| : \eta \in \widehat{C^*_r(G)}\} \geq |\chi(\tilde{a})| = |h(\omega)(a)|.$$

We will therefore show that $\|\tilde{a}\|_r = \|a\|_r$. To that end, given $\xi \in \ell^2(G)$, define $\tilde{\xi}$ by

$$\tilde{\xi}_h = \chi(u_h^{-1})\overline{\omega(h)}\xi_h.$$

Since u_h is a unitary for each $h \in G$, and χ is a $*$-homomorphism, it follows that $\|\tilde{\xi}\|_2^2 = \|\xi\|_2^2$. Moreover,

$$\lambda(\tilde{a})\tilde{\xi}(g) = \sum_{k \in G} a_k \omega(k)\overline{\chi(u_k)}\tilde{\xi}_{k^{-1}g} = \sum_k a_k \omega(k)\overline{\chi(u_k)}\chi(u_{g^{-1}k})\overline{\omega(k^{-1}g)}\xi_{k^{-1}g},$$

and since both χ and ω are multiplicative, we see that

$$\lambda(\tilde{a})\tilde{\xi}(g) = \omega(g)\chi(u_g^{-1})\sum_k a_k \xi_{k^{-1}g} = \omega(g)\chi(u_g^{-1})(\lambda(a)\xi)(g).$$

As $|\omega(g)| = |\chi(u_g^{-1})| = 1$, we have $\|\lambda(\tilde{a})\tilde{\xi}\|_2^2 = \|\lambda(a)\xi\|_2^2$. It follows that

$$\|\tilde{a}\|_r \leq \sup\{\|\lambda(\tilde{a})\tilde{\xi}\|_2 : \|\xi\|_2 = 1\} = \sup\{\|\lambda(a)\xi\|_2 : \|\xi\|_2 = 1\} = \|a\|_r.$$

(A symmetric argument shows the other inequality, so that $\|\tilde{a}\|_r = \|a\|_r$.) In other words,

$$|h(\omega)a| \leq \|\tilde{a}\|_r = \|a\|_r,$$

so our formula for $h(\omega)$ determines an element of $\widehat{C^*_r(G)}$ as claimed.

The fact that h is continuous is a fairly straightforward argument using the definition of the weak-$*$ topology. Suppose $(\omega_i)_{i\in\Lambda} \subseteq \widehat{G}$ is Cauchy. We need to see that $(h(\omega_i))_{i\in\Lambda}$ is Cauchy, i.e. we need to show that for any $a \in C^*(G)$ the net $(h(\omega_i)(a))_{i\in\Lambda} \subseteq \mathbb{C}$ is Cauchy. If $a \in \mathbb{C}G$, so that $a = \sum_{g\in G} a_g u_g$ and $a_g = 0$ for all but finitely many g, choose

$$\varepsilon < \frac{1}{|\{g : a_g \neq 0\}|}\min\{\frac{1}{|a_g|} : a_g \neq 0\}.$$

Since $(\omega_i)_{i\in\Lambda}$ is Cauchy, and $a_g \neq 0$ for only finitely many g, we can choose I such that if $i, j \geq I$ then

$$|\omega_i(g) - \omega_j(g)| < \varepsilon \text{ whenever} a_g \neq 0.$$

For $i, j \geq I$, we have $|h(\omega_i)(a) - h(\omega_j)(a)| < \varepsilon$.

If $a \in C^*(G)$ is the limit of a sequence $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{C}G$, then an $\varepsilon/3$ argument and the fact that each $h(\omega_i)$ is norm-decreasing will tell us that again, $(h(\omega_i)(a))_{i\in\Lambda}$ is Cauchy. It follows that $(h(\omega_i))_{i\in\Lambda}$ is Cauchy, as desired.

Checking that h is bijective is also straightforward. Given $\phi \in \widehat{C^*_r(G)}$, define $\omega_\phi : G \to \mathbb{C}$ by

$$\omega_\phi(g) := \phi(u_g).$$

Observe first that since ϕ is a $*$-homomorphism, $\phi(u_g) \in \mathbb{T}$ for all g, so in order to show that $\omega \in \widehat{G}$ we only need to show that ω is multiplicative. But this follows immediately from the fact that ϕ is a $*$-homomorphism:

$$\omega_\phi(g)\omega_\phi(h) = \phi(u_g)\phi(u_h) = \phi(u_g u_h) = \phi(u_{gh}) = \omega_\phi(gh).$$

It is similarly immediate to check that for a fixed $\omega \in \widehat{G}$, $\omega_{h(\omega)} = \omega$, and that $h(\omega_\phi) = \phi$. It follows that $\omega \mapsto h(\omega)$ is a bijection.

Finally, we conclude the proof by showing that the inverse function $h^{-1} : \widehat{C^*_r(G)} \to \widehat{G}$, given by $h^{-1}(\phi) = \omega_\phi$, is continuous. Suppose that $(\phi_i)_i \subseteq \widehat{C^*_r(G)}$ is Cauchy – that is, for any $a \in C^*_r(G)$ the net $(\phi_i(a))_i \subseteq \mathbb{C}$ is Cauchy. In particular, the net

$$(\phi_i(u_g))_i = (\omega_{\phi_i}(g))_i \subseteq \mathbb{T}$$

is Cauchy for each $g \in G$. By definition, then, h^{-1} is continuous. □

Corollary 5.26. *If G is a discrete abelian group, then $C^*(G) \cong C^*_r(G)$.*

6. Graph C*-algebras

Graph C*-algebras and their generalizations are a very active area of current research. They are a nifty way to build and study examples of C*-algebras, because the structure of the graph C*-algebra is very closely tied to the structure of the underlying graph. That said, the formal definition of a graph C*-algebra may seem a bit strange if you're seeing it for the first time. If you're familiar with the Cuntz and Cuntz–Krieger algebras, the definition of graph algebras will seem more natural.

Definition 6.1. Let $E = (E^0, E^1, r, s)$ be a directed graph, with vertices E^0, edges E^1, and range and source maps $r, s : E^1 \to E^0$. Both E^0 and E^1 may be infinite, but we require that E be *row-finite*. That is, for all $v \in E^0$ we require that

$$vE^1 := \{e \in E^1 : r(e) = v\} \text{ is finite.}$$

The *graph* C*-*algebra* $C^*(E)$ is the universal C*-algebra generated by a set of projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ such that:

(1) $\{p_v : v \in E^0\}$ is a set of mutually orthogonal projections: $p_v p_w = \delta_{v,w} p_v$.
(2) For any edge e, $s_e^* s_e = p_{s(e)}$.
(3) For any vertex v with $r^{-1}(v) \neq \emptyset$, we have $p_v = \sum_{r(e)=v} s_e s_e^*$.

Relations (1)–(3) are often called the *Cuntz–Krieger relations*, because of the close link between graph C*-algebras and Cuntz–Krieger algebras.

If you recall that the *source projection* of a partial isometry w is w^*w, it becomes easier to keep conditions (2) and (3) straight. Condition (2) says that the source projection of the partial isometry s_e is the projection at the source of e, while Condition (3) says that the range projections of all the edges with range v add up to p_v.

Remark 6.2. The projections and partial isometries used to construct $C^*(E)$ are purely abstract objects – you should not think of them as living on/in/near the graph E. Rather, to build $C^*(E)$, we take the graph E as inspiration (see Exerise 6.4 below), but we construct $C^*(E)$ in our imaginations, using the algebraic characterizations of projections ($p = p^2 = p^*$) and partial isometries ($s = ss^*s$) and the Cuntz–Krieger relations, and requiring the resulting abstract object to be a C*-algebra.

What do we mean by "univeral C*-algebra" in Definition 6.1? The meaning is similar to the meaning of the universal group C*-algebra. To be precise: By "the universal C*-algebra," we mean that $C^*(E)$ is the unique C^*-algebra such that, whenever $\{P_v, S_e : v \in E^0, e \in E^1\} \subseteq B(\mathcal{H})$ is any collection of projections and partial isometries satisfying (1)–(3) of Definition 6.1, there is a surjective $*$-homomorphism $C^*(E) \to C^*(\{P_v, S_e\}_{v,e})$ which sends p_v to P_v and s_e to S_e. (Compare with Proposition 5.10 in the group case.)

How do we know that $C^*(E)$ always exists, and why can we claim that a unique one exists? For existence, we construct a very big representation of $C^*(E)$; see [17, Proposition 1.20]. For uniqueness, if A is another C*-algebra with the same universal property as $C^*(E)$, then the universal properties of $C^*(E)$ and A (and the fact that both satisfy the Cuntz–Krieger relations) imply that we have mutually inverse $*$-homomorphisms $\phi : A \to C^*(E)$ and $\psi : C^*(E) \to A$, so $A \cong C^*(E)$.

Remark 6.3. There are two conventions in the literature for describing graph C*-algebras. The other one requires $s_e^* s_e = p_{r(e)}$ and $p_v = \sum_{s(e)=v} s_e s_e^*$. We've chosen this convention, not only because it's what the lecturers use in their research and will thus avoid driving them batty, but also because the mnemonic given above for distinguishing (2) and (3) doesn't work in the alternative convention. Because of the way sources and ranges switch roles between the two conventions, one can usually translate from one convention to the other by flipping the directions of all of the edges.

One consequence of this is that in our convention, paths point right-to-left. That is,

$$ef \text{ is a well-defined path in E} \iff s(e) = r(f).$$

Exercise 6.4. For edges e, f in E, when is $s_e s_f \in C^*(E)$ nonzero? Prove that $s_e s_f$ is always a partial isometry. Compare the range and source projections of $s_e s_f$, s_e, and s_f.

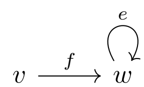
Exercise 6.5. Given any two edges e, f in E, we have $s_e s_e^* s_f s_f^* = \delta_{e,f} s_e s_e^*$ in $C^*(E)$. More generally, $s_e^* s_f = \delta_{e,f} p_{s(e)}$. Conclude that

$$C^*(E) = \overline{\operatorname{span}}\{s_\lambda s_\mu^* : \lambda = e_1 \cdots e_\ell,\ \mu = f_1 \cdots f_m \text{ are paths in } E\}.$$

Enough theorizing (for a moment). Let's look at some examples of graph C*-algebras.

Example 6.6. Consider the graph E with one vertex and one edge. Note that $s_e^* s_e = s_e s_e^* = p_v$. In fact, since s_e is a partial isometry, we have $s_e p_v = p_v s_e = s_e$, so p_v acts as an identity in $C^*(E)$. The universal property of $C^*(E)$ therefore means that $p_v = 1$ and s_e is a unitary. In other words, $C^*(E)$ is the universal C*-algebra generated by a unitary: $C^*(E) \cong C(\mathbb{T}) \cong C^*(\mathbb{Z})$.

Example 6.7. Consider the following graph E:



In $C^*(E)$, we have $1 = p_v + p_w = s_f^* s_f + s_e^* s_e$. Notice that (again by Exercise 6.5) $s_e + s_f$ is an isometry:

$$(s_e + s_f)^*(s_e + s_f) = s_e^* s_e + s_f^* s_f = 1.$$

However, its range projection is

$$(s_e + s_f)(s_e + s_f)^* = s_e s_e^* + s_f s_f^* + s_e s_f^* + s_f s_e^* = p_w,$$

since $s_e s_f^* = s_e s_e^* s_e s_f^* s_f s_f^* = s_e p_w p_v s_f^* = 0$. So $s_e + s_f$ is a proper isometry.

In fact, we can recover all of the generators of $C^*(E)$ from the single isometry $s_e + s_f$. We found 1 and p_w above, so $p_v = 1 - p_w \in C^*(s_e + s_f)$. Moreover, $s_e = (s_e + s_f)p_w$ and $s_f = (s_e + s_f)p_v$. That is, $C^*(E) = C^*(s_e + s_f)$ is a universal C*-algebra generated by a non-unitary isometry. That means $C^*(E) \cong \mathcal{T}$ is the *Toeplitz algebra*, the universal C*-algebra generated by a non-unitary isometry. (You can also realize $\mathcal{T}$ as $C^*(S)$, the C*-algebra generated by the unilateral shift on ℓ^2.)

Example 6.8. Consider the graph

$$v_0 \xleftarrow[e_1]{} v_1 \xleftarrow[e_2]{} v_2 \xleftarrow[e_3]{} v_3 \cdots$$

Notice that for any i we have $s_{e_i} s_{e_i}^* = p_{v_{i-1}}$ and $s_{e_i}^* s_{e_i} = p_{v_i}$. From this, you can compute that for any $j \geq i$,

$$(s_{e_i} s_{e_{i+1}} \cdots s_{e_j})(s_{e_i} s_{e_{i+1}} \cdots s_{e_j})^* = p_{v_{i-1}}.$$

If $\ell \geq i$, write $\lambda_{i,\ell} := e_i \cdots e_\ell$, and for any $i, j \in \mathbb{N}$, define

$$s_{i,j} = s_{\lambda_{i,\max\{i,j\}}} s^*_{\lambda_{j,\max\{i,j\}}}.$$

Observe (it's a good **exercise** to check this) that $\{s_{i,j}\}_{i,j\in\mathbb{N}}$ is a family of matrix units in $C^*(E)$:

$$s_{i,j}^* = s_{j,i}, \quad s_{i,j} s_{p,q} = \delta_{j,p} s_{i,q}.$$

Therefore (cf. [17, Corollary A.9]) $C^*(\{s_{i,j}\}) \cong \mathcal{K}(\ell^2)$.

In fact, $C^*(E) \cong \mathcal{K}(\ell^2)$! We'll show this by convincing ourselves that $C^*(E) \cong C^*(\{s_{i,j}\})$. It's relatively easy to see that all of the vertex projections in $C^*(E)$ are of the form $s_{i,j}$ for some i, j. But we also have that

$$s_{e_i} = s_{e_i} s_{e_i}^* s_{e_i} = s_{e_i} p_{v_i} = s_{e_i} s_{e_{i+1}} s_{e_{i+1}}^* = s_{i,i+1} \in C^*(\{s_{i,j}\}).$$

Thus $C^*(E) \subseteq C^*(\{s_{i,j}\})$, and since $C^*(\{s_{i,j}\}) \subseteq C^*(E)$ by construction, they must be equal.

Example 6.9. Fix an integer $n \geq 2$. Consider the graph E with one vertex v and n edges $e_1, \dots, e_n$. As in Example 6.6 and 6.7, we can compute that $p_v = 1 \in C^*(E)$. Thus, the generators $s_i := s_{e_i}$ satisfy

$$s_i^* s_i = 1 \ \forall\ i, \qquad \sum_{i=1}^n s_i s_i^* = 1.$$

That is, $C^*(E)$ is the universal C*-algebra generated by n isometries $\{s_i\}_{i=1}^n$ satisfying $\sum_i s_i s_i^* = 1$. Another name for this C*-algebra is the *Cuntz algebra* $\mathcal{O}_n$. These were invented by Joachim Cuntz [1977] in order to provide examples of simple separable infinite C*-algebras. The Cuntz algebras are deceptively complicated and interesting C*-algebras; there's lots of literature about them.

A lot of the structure of graph C*-algebras can be determined directly from the graph, such as the K-theory of $C^*(E)$ and its lattice of ideals.

Definition 6.10. Let E be a directed graph. For vertices $v, w \in E^0$, we write

$$w \leq v \iff \text{there is a (directed) path in } E \text{ with source } v \text{ and range } w.$$

In symbols, this is summarized by $wE^*v \neq \emptyset$.

In a directed graph E, a subset $H \subseteq E^0$ is *hereditary* if whenever $w \in H$ and $w \leq v$ then $v \in H$. A subset $H \subseteq E^0$ is *saturated* if

$$\emptyset \neq s(r^{-1}(v)) \subseteq H \implies v \in H.$$

Proposition 6.11. *Let E be a row-finite directed graph. If $I \subseteq \mathrm{C}^*(E)$ is a nonzero ideal, then the set*

$$H_I = \{v \in E^0 : p_v \in I\}$$

is hereditary and saturated.

Proof. Suppose $v \in H_I$ and $v \leq w$. That is, there is a path $e_1 e_2 \cdots e_n$ in E with $s(e_n) = w$ and $r(e_n) = v$. By the third Cuntz–Krieger relation, if $2 \leq i \leq n$, we have

$$p_{s(e_{i-1})} s_{e_i} = \left(\sum_{e \in s(e_{i-1})E^1} s_e s_e^* \right) s_{e_i} = s_{e_i} s_{e_i}^* s_{e_i} + \left(\sum_{e_i \neq e \in s(e_{i-1})E^1} s_e s_e^* s_{e_i} \right) = s_{e_i};$$

the sum $\left(\sum_{e_i \neq e \in s(e_{i-1})E^1} s_e s_e^* s_{e_i} \right)$ is zero by Exercise 6.5. It follows that

$$p_w = s_{e_n}^* s_{e_n} = s_{e_n}^* p_{s(e_{n-1})} s_{e_n} = s_{e_n}^* s_{e_{n-1}}^* s_{e_{n-1}} s_{e_n} = \cdots = (s_{e_1 e_2 \cdots e_n})^* s_{e_1 \cdots e_n} = (s_{e_1 e_2 \cdots e_n})^* p_v s_{e_1 \cdots e_n}.$$

That is, if $p_v \in I$ then $p_w \in I$ as well, so H_I is hereditary.

To see that H_I is saturated, suppose that $\emptyset \neq r^{-1}(v)$ and that whenever $r(e) = v$ then $s(e) \in H_I$. Equivalently, $p_{s(e)} \in I$ and consequently $s_e = s_e p_{s(e)} \in I$ for all $e \in vE^1$. Since I is $*$-closed, we conclude that

$$p_v = \sum_{e \in vE^1} s_e s_e^*$$

is a sum of elements of I. Hence $p_v \in I$ and $v \in H_I$. □

In fact, we have the following:

Theorem 6.12. *Let E be a row-finite[10] directed graph which has no cycles. Then the map $I \mapsto H_I$ is a bijection between the ideals of $\mathrm{C}^*(E)$ and the hereditary saturated subsets of E^0. The inverse is given by $H \mapsto I_H := \langle \{p_v : v \in H\} \rangle$.*

For a proof (of a stronger result, in fact), see [17, Theorem 4.9].

We also have a nifty characterization of when a graph C*-algebra is simple.

Definition 6.13. A *cycle* in a directed graph E is a composable path $e_1 \cdots e_n$ with $r(e_1) = s(e_n)$. A cycle $e_1 \cdots e_n$ *has an entrance* if there exists $1 \leq i \leq n$ such that $r^{-1}(r(e_i)) \neq \{e_i\}$. (Draw a picture!) E is *cofinal* if for every one-sided infinite path $x = e_1 e_2 e_3 \cdots$ in E, and every vertex $v \in E^0$, there exists a path μ in E with $r(\mu) = v$ and $s(\mu) = r(e_i)$ for some i. A graph E is *source-free* if for every $v \in E^0$ we have $s^{-1}(v) \neq \emptyset$.

Theorem 6.14. *If E is a row-finite source-free graph, then $\mathrm{C}^*(E)$ is simple iff E is cofinal and every cycle has an entrance.*

We won't give the proof here, but you can read it in [17, Theorem 4.14].

The proof of Theorem 6.14 relies on the *Cuntz–Krieger Uniqueness Theorem:*

Theorem 6.15. *Let E be a row-finite graph in which every cycle has an entrance. If $\{P_v, S_e : v \in E^0, e \in E^1\} \subseteq B(\mathcal{H})$ satisfies the Cuntz–Krieger relations for E, and $P_v \neq 0$ for all v, then $\mathrm{C}^*(E) \cong \mathrm{C}^*(\{P_v, S_e\}_{v,e})$.*

For a proof, see [17, Chapter 3].

[10] We need this assumption to make sense of the third Cuntz–Krieger relation, so it's a standard (although not 100% necessary) assumption on graph C*-algebras.

7. The Gelfand-Naimark-Segal (GNS) Theorem

Preview of Lecture: In lecture, we won't discuss the proofs of the technical results we'll need about states for this lecture (eg Lemmas 7.8 and 7.12). However, these are important both for von Neumann algebraic applications and for C*-algebras, so you should read the proofs carefully and ask questions in office hours if you're confused.

We will prove Theorem 7.9 in lecture as well as Theorem 7.1. We'll discuss irreducible representations but, depending on time, perhaps not the proof of Proposition 8.5. We will, however, discuss the proof of Proposition 5.15.

There are a lot of exercises in this section! If there's time, we'll discuss a few in lecture (so please let us know if there are any that you'd particularly like to see).

The main goal of this section is the following theorem:

Theorem 7.1 (Gelfand-Naimark)**.** *Every* C**-algebra A admits a faithful nondegenerate representation* $\pi : A \to B(\mathcal{H})$*. If A is separable, π can be chosen to be separable.*

As an immediate corollary, every C*-algebra A is isomorphic to a norm-closed $*$-subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. (It can be useful to take this as the definition of a C*-algebra, which justified our using the term "C*-algebra" for abstract (not concretely represented) C*-algebras.)

Throughout this section, we generally assume A is unital, for simplicity; the arguments can all be made in general, by taking some care with approximate identities. See Exercise 7.16 below (and Remark 7.17 for why we can't just unitize our way out of this one).

Our first step on the road to proving Theorem 7.1 has to do with *states*.

Definition 7.2. A *state* on a C*-algebra A is a linear functional $\phi : A \to \mathbb{C}$ which is *positive* in that $\phi(a) \geq 0$ whenever $a \geq 0$, and such that

$$\|\phi\| := \sup\{|\phi(a)| : \|a\| = 1\} = 1.$$

The subset $\mathcal{S}(A) \subset A^*_{\leq 1}$ consisting of states is called the *state space*.

Example 7.3. If π is a representation of A on $\mathcal{H}$, and $\xi \in \mathcal{H}$ has norm 1, the function

$$\phi(a) := \langle \pi(a)\xi, \xi\rangle$$

is a state on A.

Example 7.4. A character $\phi : A \to \mathbb{C}$ is both a state and a representation. (**check**)

Exercise 7.5. Show that any positive linear functional $\phi : A \to \mathbb{C}$ is $*$-preserving, i.e. $\phi(a^*) = \overline{\phi(a)}$ for all $a \in A$.

Exercise 7.6. Show that $\mathcal{S}(A)$ is a weak*-closed convex subset of $A^*_{\leq 1}$. It follows from Alaoglu's theorem that it is weak*-compact. What does the Krein-Milman theorem say about $\mathcal{S}(A)$?

Given a state[11] ϕ on A, if we define $[a, b]_\phi := \phi(b^*a)$, then this form[12] on A is positive sesquilinear (linear in the first variable, conjugate linear in the second variable plus $[a, a]_\phi \geq 0 \ \forall\ a \in A$) and hence satisfies the Cauchy-Schwarz inequality:

Exercise 7.7. Show that $|[a, b]_\phi|^2 \leq [a, a]_\phi\, [b, b]_\phi = \phi(a^*a)\phi(b^*b)$.

Here are a few facts about states that we will need later.

Lemma 7.8. *Let A be a unital* C**-algebra.*

(1) If ϕ is a state on A, then $\phi(1) = 1$.
(2) If ϕ is a bounded linear functional on A which satisfies $1 = \|\phi\| = \phi(1)$, then ϕ is a state.

---

[11]Actually, all you need is a positive linear functional for the following assertions and exercise.
[12]Sometimes this is defined by $\phi(a^*b)$. This changes which side is linear and which is conjugate linear.

Proof. (1) If ϕ is a state, then $|\phi(1)| = \phi(1) \leq \|\phi\| = 1$. For the other inequality, taking $b = 1$ in Exercise 7.7 tells us that

$$|\phi(a)|^2 \leq \phi(1)\phi(a^*a).$$

Moreover, the functional calculus (Exercise 3.11) implies that $\|a^*a\|1_A \geq a^*a$ for all $a \in A$. Consequently $\phi(a^*a) \leq \|a^*a\|\phi(1)$. It follows that for any $a \in A$ with $\|a\| = 1$,

$$|\phi(a)|^2 \leq \phi(1),$$

and hence $\|\phi\| \leq \phi(1)$. We conclude that $1 = \|\phi\| = \phi(1)$.

(2) If $\phi \in A^*$ with $1 = \phi(1) = \|\phi\|$, then once we know that ϕ is positive, ϕ must be a state. First we claim that $\phi(a) \in \mathbb{R}$ for any $a \in A_{s.a.}$. Pick $a \in A_{s.a.}$ and write $\phi(a) = \alpha + i\beta$. By possibly replacing a with $-a$, we may assume that $\beta \geq 0$. We claim that $\beta = 0$. Fix $n \in \mathbb{N}$ and observe that, since $a^* = a$,

$$\|n - ia\|^2 = \|(n + ia^*)(n - ia)\| = \|n^2 + in(a^* - a) + a^2\| \leq= \|n^2 + a^2\| \leq n^2 + \|a^2\| = n^2 + \|a\|^2.$$

On the other hand, $|\phi(n - ia)|^2 = |n\phi(1) - i\alpha + \beta|^2 = (n + \beta)^2 + \alpha^2$. So,

$$(n^2 + \|a\|^2) = \|\phi\|^2(n^2 + \|a\|^2) \geq \|\phi\|^2 \, \|n - ia\|^2 \geq |\phi(n - ia)|^2 = n^2 + 2n\beta + \beta^2 + \alpha^2.$$

In order for this inequality to hold for all $n \in \mathbb{N}$ we must have $\beta = 0$, as claimed. Hence $\phi(a) \in \mathbb{R}$.

Now fix a positive element in A_+, which we also call a, and assume $\|a\| \leq 1$. Then Proposition 3.6 implies that $\|1 - a\| \leq 1$. Since $\|\phi\| = 1$ by hypothesis,

$$1 \geq \|1 - a\| \geq \phi(1 - a) = \phi(1) - \phi(a) = 1 - \phi(a).$$

Since $\phi(a) \in \mathbb{R}$, it follows that $\phi(a) \geq 0$ for any positive a. □

The next theorem is the cornerstone of our proof of Theorem 7.1.

Theorem 7.9 (GNS construction)**.** *If ϕ is any state on a* C*-*algebra A, there is a nondegenerate representation $\pi_\phi : A \to B(\mathcal{H})$ and a unit vector $\xi \in \mathcal{H}$ such that $\phi(a) = \langle \pi_\phi(a)\xi, \xi\rangle$ for any $a \in A$.*

The representation π_ϕ is called the *GNS representation associated to ϕ*, and the vector ξ is called a *cyclic vector* for the representation π_ϕ.

Proof of unital case. Assume A is unital. We will build $\mathcal{H}$ out of A itself. Recall that $[\cdot,\cdot]_\phi$ is a positive sesquilinear form. The only thing standing between us and an inner product are the nonnegative elements in A with $\phi(a^*a) = 0$. Let's mod them out.

Set $N_\phi = \{a \in A : [a,a]_\phi = 0\}$. Observe (check!) that N_ϕ is a vector subspace of A, which is closed in norm. (The fact that N_ϕ is closed under addition follows from Exercise 7.7. Proving that N_ϕ is closed in norm is also a good exercise.)

Let X be the vector space quotient $X = A/N_\phi$, and define an inner product on X by

$$\langle a + N_\phi, b + N_\phi\rangle_\phi := \phi(b^*a).$$

To see that this is well defined, first note that Exercise 7.7 tells us that

$$N_\phi = \{a \in A \mid \phi(b^*a) = 0, \ \forall\ b \in A\}.$$

It follows that for any $x, y \in N_\phi$ and $a, b \in A$,

$$\phi((b+y)^*(a+x)) = \phi(b^*a) + \phi(b^*x) + \phi(y^*a) + \phi(y^*x) = \phi(b^*a) + \phi(b^*x) + \overline{\phi(a^*y)} + \phi(y^*x) = \phi(b^*a).$$

Take $\mathcal{H}_\phi$ to be the completion of X with respect to the norm induced by $\langle\cdot,\cdot\rangle_\phi$. Now we have a Hilbert space. Our representation $\pi_\phi : A \to B(\mathcal{H}_\phi)$ is given by left multiplication: $\pi_\phi(a)(b + N_\phi) = ab + N_\phi$.

First, we need to check that this map is well defined in the sense that $\pi_\phi(a)$ is a bounded linear operator for all a.

Since $a^*a \in A$ is positive, Exercise 3.11 (2) tells us that $\|a^*a\|1 - a^*a \geq 0$. Thus, for any $x \in A$, Exercise 3.11 (1) tells us that

$$0 \leq x^*(\|a^*a\|1 - a^*a)x = \|a^*a\|x^*x - x^*a^*ax,$$

so $(ax)^*ax \leq \|a^*a\| x^*x$. In particular,

$$\begin{aligned}\|\pi_\phi(a)\|^2 &= \sup\{\langle \pi_\phi(a)(x+N_\phi), \pi_\phi(a)(x+N_\phi)\rangle_\phi \mid x \in A,\ \phi(x^*x)=1\}\\ &= \sup\{\phi((ax)^*(ax)) \mid x \in A,\ \phi(x^*x)=1\}\\ &\leq \sup\{\phi(\|a^*a\|x^*x) \mid x \in A,\ \phi(x^*x)=1\}\\ &\leq \|a^*a\| = \|a\|^2.\end{aligned}$$

So, we conclude that $\pi_\phi(a) \in B(\mathcal{H}_\phi)$ for all $a \in A$. We leave it as an **exercise** to check that π_ϕ is a *-homomorphism.

Since π_ϕ is unital (why?), it is nondegenerate. Finally, note that the unit vector ξ such that $\phi(a) = \langle \pi(a)\xi, \xi\rangle$ is $\xi = 1 + N_\phi$. □

Exercise 7.10. Check that π_ϕ is a *-homomorphism. (In checking that π_ϕ is $*$-preserving, you will see why we defined $\langle \cdot, \cdot \rangle_\phi$ as we did.)

Remark 7.11. If you ever see in some proof in the literature a representation unceremoniously associated to some state (in particular for a trace that is moreover a state), it's assumed to be the GNS representation constructed as above.

To prove Theorem 7.1, we will take the direct sum of a lot of the representations whose existence we have just established.

Lemma 7.12. *Let A be a* C**-algebra and a a nonzero normal element of A. Then there is a state ψ on A such that $|\psi(a)| = \|a\|$.*

Proof. Let $B = C^*(\{a, 1\}) \subseteq \widetilde{A}$. Fix $\lambda \in \sigma(a)$ with $|\lambda| = r(a)$ maximal, and let $\mathrm{ev}_\lambda : C(\sigma(a)) \to \mathbb{C}$ be given by evaluation at λ. Recall that ev_λ is a character on $C(\sigma(a))$ and hence also a state on $C(\sigma(a))$.

Since B is a closed subspace of $\widetilde{A}$, the Hahn-Banach theorem allows us to extend it to a linear functional $\psi \in \widetilde{A}^*$ with the same norm (i.e., $\|\psi\| = 1$). As $\psi(1) = \mathrm{ev}_\lambda(1) = 1$, Lemma 7.8 tells us that ψ is also a state. Furthermore, as the Gelfand transform $\Gamma : B \overset{\cong}{\to} C(\sigma(a))$ takes a to the function $f(z) = z$, it follows that $|\psi(a)| = |\lambda| = r(a) = \|a\|$ (since a is normal). □

Corollary 7.13. *If $F \subseteq \mathcal{S}(A)$ is a subset of the states of A which is dense in the weak-$*$ topology, then for any $a \in A$,*

$$\sup\{|\phi(a)| : \phi \in F\} = \|a\|.$$

We are finally ready to prove what is often called the Gelfand-Naimark Theorem.

Proof of Theorem 7.1. Choose a subset F of $\mathcal{S}(A)$ which is dense in the weak-$*$ topology on $\mathcal{S}(A) \subseteq A^*$. Define $\pi := \bigoplus_{\phi \in F} \pi_\phi$, where π_ϕ is the representation arising from the state ϕ as in Theorem 7.9.

Fix $a \in A$. Since $\phi(1) = 1$,

$$\|\pi(a)\|^2 = \sup_{\phi \in F} \|\pi_\phi(a)\|^2 = \sup_{\phi \in F} \sup\{\langle \pi_\phi(a^*a)\xi, \xi\rangle : \xi \in \mathcal{H}_\phi\} \geq \sup_{\phi \in F} \langle \pi_\phi(a^*a)1, 1\rangle = \sup_{\phi \in F} \phi(a^*a) = \|a\|^2.$$

As π is a $*$-homomorphism and therefore norm-decreasing, it follows that $\|\pi(a)\| = \|a\|$ for all $a \in A$. Since π is unital, it is nondegenerate.[13]

If A is separable, then [6, Theorem V.5.1] implies that $A^* \supseteq \mathcal{S}(A)$ is too, so we can take the set F to be countable. The separability of A implies the separability of $\mathcal{H}_\phi$ for each ϕ,[14] so $\mathcal{H}$ is separable. □

Definition 7.14. The representation

$$\pi_u := \bigoplus_{\phi \in \mathcal{S}(A)} \pi_\phi : A \to B\left(\bigoplus_{\phi \in \mathcal{S}(A)} \mathcal{H}_\phi\right) =: B(\mathcal{H}_u)$$

is called the *universal representation* of A.

[13] The fact that π is nondegenerate also follows from the fact that each π_ϕ is nondegenerate, which in turn follows from our construction of $\mathcal{H}_\phi$ as a completion of (a quotient of) A.

[14] There is something to check here, since the norm on $\mathcal{H}_\phi$ is not the same as the norm on A. **Exercise:** How do they relate?

Remark 7.15. For the sake of a faithful representation, we could instead form the direct sum over a weak*-dense subset of $\mathcal{S}(A)$. However, we often really want to take *all* the states and get π_u because the associated von Neumann algebra $\pi_u(A)''$ is isometrically isomorphic to A^{**}. Both are often called the *enveloping von Neumann algebra* of A.

Exercise 7.16. Generalize the results in this section to non-unital C*-algebras. (In particular, you will have to show that an approximate unit $(e_\lambda)_\lambda$ becomes Cauchy in A/N_ϕ for any state ϕ, and hence gives rise to a cyclic vector in any GNS representation π_ϕ.)

To check that the resulting representation is still nondegenerate, consider in more detail what happens in the unital case:

Suppose that $\pi_\phi(a)(x + N_\phi) = 0$ for all a. In particular, taking $a = 1$,

$$0 = \|\pi_\phi(1)(x + N_\phi)\|^2 = \langle x + N_\phi, x + N_\phi\rangle_\phi = \phi(x^*x),$$

so we must have $x \in N_\phi$.

Remark 7.17. Why can't we just unitize in Exercise 7.16? Well, as easy as it was to always guarantee a unique extension of a $*$-homomorphism to the unitization, it is no longer true in general for positive linear maps. (We'll return to this in Proposition 10.22.) It turns out this *is* true for states, i.e., they extend to states on $\tilde{A}$. However, the proof of this fact will require the non-unital version of 7.9 (because it allows us to borrow from this fact for representations). The proof of this fact takes us a little off course, so we will state it here with reference:

[8, Corollary 1.9.7] Every state on a nonunital C*-algebra A extends uniquely to a state on $\tilde{A}$.

What does Theorem 7.1 say about Abelian C*-algebras? In this case, the Riesz-Markov-Kakutani representation theorem tells us that states on $C_0(X)$ are in bijection with probability measures on X, so that $\phi(f) = \int_X f\, d\mu_\phi$. Note that N_ϕ consists of the set of C_0 functions on X which are 0 off a μ_ϕ-null set. Thus, $\mathcal{H}_\phi = \overline{C_0(X)/N_\phi} \cong L^2(X, \mu_\phi)$, and π_ϕ represents $C_0(X)$ on $L^2(X, \mu_\phi)$ as multiplication operators:

$$\pi_\phi(f)\xi = x \mapsto f(x)\xi(x).$$

To me at least, this is reminiscent of the link between the continuous and the Borel functional calculus.

Exercise 7.18. What does the universal representation of an Abelian C*-algebra look like?

Exercise 7.19. Let A be a C*-algebra.

(1) Show that for any $b \in A$, there exists a representation $\pi : A \to B(\mathcal{H})$ and unit vector $h \in \mathcal{H}$ so that $\|\pi(b)h\| = \|b\|$. (Hint: Apply Lemma 7.12 to $a = b^*b$.)
(2) Use Exercise 4.29 to give a different argument for the last claim in Theorem 7.1, i.e. that any separable C*-algebra has a faithful separable representation.

7.1. **Applications.** We've already seen the GNS theorem invoked several times, for structural results about C*-algebras. Here are some of those delayed proofs.

Exercise 7.20. Show again that if $0 \le a \le b$, then $\|a\| \le \|b\|$, without assuming a and b commute.

Exercise 7.21. Show that if the C*-algebra A is finite dimensional as a vector space, then we may take the Hilbert space $\mathcal{H}$ of Theorem 7.1 to be finite dimensional. *Hint:* Show that you only need finitely many states $\phi \in F$, and that H_ϕ is finite dimensional for all ϕ.

Exercise 7.22. Use the GNS theorem to give a very quick proof of Theorem 3.10.

Exercise 7.23. For a commutative C*-algebra A, what would a weak$*$-dense subspace of $\mathcal{S}(A)$ look like?

The proof of Proposition 5.15 relies on *positive definite functions* on groups, and their connection with states on $C^*(G)$.

Definition 7.24. Let G be a discrete group. A function $\psi : G \to \mathbb{C}$ is *positive definite* if, for any finite subset $F \subseteq G$, the matrix M^ψ in $M_F(\mathbb{C})$ given by

$$M^\psi_{s,t} = \psi(s^{-1}t)$$

is positive.

Proposition 7.25. *If ϕ is a state on $\mathrm{C}^*(G)$, then the function $\psi^\phi(g) = \langle \pi_\phi(u_g)1, 1\rangle_\phi$ is positive definite. Conversely, every positive definite function defines a state on $\mathrm{C}^*(G)$.*

Proof. If ϕ is a state on $\mathrm{C}^*(G)$, we compute that

$$M^{\psi^\phi}_{s,t} = \langle \pi_\phi(u_{s^{-1}t})1, 1\rangle = \langle \pi_\phi(u_t)1, \pi_\phi(u_s)1\rangle = \phi(u_t)\overline{\phi(u_s)}.$$

In other words, if T is the matrix with entries indexed by elements of G, such that the first column consists of the entry $\phi(u_s)$ in the sth row, and T is zero in all other columns, then $M^{\psi^\phi} = T^*T$ is positive. So ψ^ϕ is positive definite, as claimed.

For the converse, given a positive definite function ψ, define $\phi_\psi(\sum_g a_g u_g) := \frac{1}{\psi(e)} \sum_g a_g \psi(g)$. By construction, ϕ_ψ is a linear functional on $\mathbb{C}G$. Considering the set $F = \{e\}$ tells us that $\psi(e) > 0$, so ϕ_ψ is well defined, and moreover that

$$\phi_\psi(u_e) = \frac{\psi(e)}{\psi(e)} = 1. \tag{7.1}$$

Moreover, ϕ_ψ is bounded with respect to $\|\cdot\|_u$, because

$$|\phi_\psi(\sum_g a_g u_g)| = |\langle \pi_{\phi_\psi}(\sum_g a_g u_g)1, 1\rangle| \le \|\pi_{\pi_\psi}(\sum_g a_g u_g)\| \le \|\sum_g a_g u_g\|_u. \tag{7.2}$$

It now follows that $\|\phi_\psi\| = 1$: equation (7.1) implies that $\|\phi_\psi\| = \sup\{|\phi_\psi(f)| : \|f\|_u = 1\} \ge |\phi_\psi(u_e)| = 1$, and equation (7.2) implies that $\|\phi_\psi\| \le 1$. Thus, Lemma 7.8(2) tells us that ϕ_ψ extends to a state on $\mathrm{C}^*(G)$. ☐

Proof of Proposition 5.15. We first address the case of the reduced C*-algebras. Suppose $G \le H$ are discrete groups, and decompose $\ell^2(H) = \bigoplus_h \ell^2(Gh)$ via the right cosets of G. Notice that the left regular representation of $\mathbb{C}G \subseteq \mathbb{C}H$ on $\ell^2(H)$ preserves this decomposition, and $\ell^2(Gh) \cong \ell^2(G)$ (via a canonical isomorphism) for any $h \in H$. As the operator norm of a direct sum satisfies

$$\|f \oplus g\| = \max\{\|f\|, \|g\|\},$$

it follows that the norm induced on $\mathbb{C}G$ by the left regular representation λ^H is the same as the norm induced by λ^G. In other words, the inclusion $\mathbb{C}G \subseteq \mathbb{C}H$ is isometric with respect to the reduced norm, so $\mathrm{C}^*_r(G) \subseteq \mathrm{C}^*_r(H)$.

Now, we show that if $G \le H$ (and G is countable) then $\mathrm{C}^*(G) \le \mathrm{C}^*(H)$. The fact that G countable implies that $\mathrm{C}^*(G)$ is separable. In this case,$\mathrm{C}^*(G)^*$ is also separable, so there exists a faithful state ϕ on $\mathrm{C}^*(G)$: namely, for a weak-$*$ dense subset $\{\omega_n\}_{n\in\mathbb{N}}$ of $\mathcal{S}(A)$, take $\phi = \sum_n 2^{-n}\omega_n$. It is straightforward to check that, thanks to the density of $\{\omega_n\}_n$, $\phi(a) = 0$ implies $a = 0$, so ϕ is indeed faithful.

Consider the positive definite function ψ^ϕ on G which Proposition 7.25 associates to ϕ. Extend it to ψ on H by setting $\psi(h) = 0$ whenever $h \notin G$. To see that ψ is positive definite, note first that if $s, t \in H$ and $sG \ne tG$, then $s^{-1}t \notin G$ and therefore $M^\psi_{s,t} = 0$. In other words, for any finite set F, M^ψ is block diagonal, where each block is indexed by $F \cap sG$ for a single *left* coset sG of G. Block diagonal matrices are positive precisely when each block is positive, so to see that ψ is positive definite it suffices to consider the matrices M^ψ associated to finite sets $F \subseteq sG$ which are contained in a single coset. For any $g, h \in G$,

$$M^\psi_{sg,sh} = \psi(g^{-1}h) = \psi^\phi(g^{-1}h) = M^{\psi^\phi}_{g,h},$$

so the fact that ψ^ϕ is positive definite implies that ψ is as well.

Now, consider the GNS representation π_ψ associated to ϕ_ψ. As ϕ_ψ and ϕ agree on $\mathrm{C}^*(G)$, it follows that for any $f \in \mathrm{C}^*(G)$,

$$\|\tilde\iota(f)\|_{u,H} \ge \|\pi_\psi(f)\| = \|\pi_\phi(f)\|.$$

The fact that ϕ is faithful means that π_ϕ is injective and therefore isometric, by Theorem 4.13: if $\pi_\phi(f) = 0$ then

$$\begin{aligned} 0 = \|\pi_\phi(f)\|^2 &= \sup\{\|\pi_\phi(f)[a]\|^2 : [a] \ \in \mathcal{H}_\phi = \overline{\mathrm{C}^*(G)}^{\|\cdot\|_\phi}, \|[a]\| = 1\} \\ &= \sup\{\phi(a^*f^*fa) : \phi(a^*a) = 1\} \ge |\phi(f^*f)| = |\phi(f)|^2 \end{aligned}$$

by Exercise 7.5. The fact that ϕ is faithful then implies that $f = 0$. In other words, $\|\pi_\phi(f)\| = \|f\|_{u,G}$ for any $f \in \mathrm{C}^*(G)$. So $\|\tilde\iota(f)\|_{u,H} \ge \|f\|_{u,G}$. As we saw in Monday's notes that $\tilde\iota : \mathrm{C}^*(G) \to \mathrm{C}^*(H)$ is norm-decreasing, it now follows that $\|\tilde{\iota}(f)\|_{u,H} = \|f\|_{u,G}$. Consequently, $\tilde{\iota}$ must be injective: if $\tilde{\iota}(f) = 0$ then $f = 0$. □

Remark 7.26. For those that are wondering whether all of this rigamarole about positive definite functions is really necessary: If you try to extend a state from $C^*(G)$ to $C^*(H)$ by just making it zero on all elements not coming from $C^*(G)$, it's hard to prove directly that this extension is still a state.

Now that we have faithful representations, we are ready to give our first proof of a powerful and useful tool in C*-algebras. Aloud we usually reference it by saying something like, "contractions lift to contractions" (with the assumption that we can scale to get the full result).

Proposition 7.27. *Let $\pi : A \to B$ be a $*$-homomorphism between C*-algebras and $b \in \pi(A)$. Show that there exists $a \in A$ with $\pi(a) = b$ and $\|a\| = \|b\|$.*

Proof. First, by possibly identifying B with its image inside $\tilde{B}$, we assume B is unital. Moreover, it suffices to show the claim for $\|b\| = 1$ (why?).

We by choosing any $a \in A$ with $\pi(a) = b$. Then $1 = \|\pi(a)\| \leq \|a\|$. If we have equality, then there is nothing to do. So, we assume $\|a\| > 1$, and hence also that its positive part $|a|$ has norm strictly greater than 1. By taking a faithful representation, we assume $A \subset B(\mathcal{H})$ and let $a = u|a|$ be the polar decomposition of a in $B(\mathcal{H})$. Define a function $f \in C[0, \|a\|]$ by

$$f(t) = \begin{cases} t & ; \quad t \in [0,1] \\ 1 & ; \quad t \in (1, \|a\|] \end{cases}$$

Note that $f(|a|) \in C^*(a) \subset C^*(a, 1_{\mathcal{H}}) \cong \tilde{A}$. Now, we define $g \in C[0, \|a\|]$ by

$$g(t) = \begin{cases} 1 & ; \quad t \in [0,1] \\ t^{-1} & ; \quad t \in (1, \|a\|] \end{cases}$$

If A is unital, then $g(|a|) \in A$, and if not, it's in $C^*(A, 1_{\mathcal{H}})$. But notice that $tg(t) = f(t)$ for all $t \in [0, \|a\|]$ with $f(|a|) \in C^*(|a|)$, and so[15]

$$uf(|a|) = u|a|g(|a|) = ag(|a|) \in C^*(a) \subset A.$$

Moreover,

$$\|uf(|a|)\| \leq \|u\|\|f(|a|)\| = \|f(|a|)\| \leq 1.$$

(**Exercise:** Why is $\|u\| = 1$?) (We could give a better estimate for the norm here, but this is all we need.) We claim $uf(|a|)$ is the desired lift of b. Indeed, let $\tilde{\pi}$ denote the unital extension of π to $C^*(A, 1_{\mathcal{H}})$ (where we take $\tilde{\pi} = \pi$ if $C^*(A, 1_{\mathcal{H}}) = A$, i.e. if A is unital). Then, by Exercise 2.33,

$$\pi(uf(|a|)) = \tilde{\pi}(uf(|a|)) = \tilde{\pi}(ag(|a|)) = \tilde{\pi}(a)g(|\tilde{\pi}(a)|) = bg(|b|).$$

But as a continuous function in $C(\sigma(|b|))$, $g = 1$. Hence $g(|b|) = 1$, and so $bg(|b|) = b$. So, $\pi(uf(|a|)) = b$ and moreover,

$$\|b\| \geq \|uf(|a|)\| \geq \|\pi(uf(|a|))\| = \|b\|.$$

□

[15] Alternatively, we know automatically from Proposition 3.19 that $uf(|a|) \in A$.

8. AF algebras

Preview of Lecture: By definition, AF algebras are inductive limits. So, before reading this section, it would probably be a very good idea to review the section about inductive limits from the Prerequisite Notes.

We will talk about, but not prove, the fact that inductive limits always exist in the category of C*-algebras. We will talk about Bratteli diagrams in lecture, probably via Example 8.13.

The three paragraphs before Exercise 8.17 are meant to provide inspiration for future reading or research; we won't discuss them in lecture.

To discuss AF algebras, the first Proposition we need is another application of the GNS construction. In order to prove that, we need to start by discussing irreducible representations of C*-algebras.

The following should rightfully be called a Definition/Theorem. The proof uses results that take us a little far afield, so we give it as a definition and refer you to [8, Lemma 1.9.1-Theorem 1.9.4] for a proof.

Definition 8.1. A representation $\pi : A \to B(\mathcal{H})$ of a C*-algebra A is *irreducible* if one of the following equivalent conditions hold:

(1) π has no proper invariant subspaces, i.e. no subspace $V \subsetneq \mathcal{H}$ so that $\pi(a)V \subset V$ for all $a \in A$.
(2) π has no proper invariant manifolds (i.e. subspaces which may or may not be closed).
(3) $\pi(A)' = \mathbb{C}1_{\mathcal{H}}$.

Under the additional assumption that π has a cyclic unit vector $h \in \mathcal{H}$, these are also equivalent to

(4) The state $a \mapsto \langle \pi(a)h, h\rangle$ is *pure*, i.e. it is an extreme point in the state space $\mathcal{S}(A)$.

Remark 8.2. We have a couple remarks on irreducible representations:

(1) First, it's sometimes helpful to see a non-example: Let $\pi_i : A \to B(\mathcal{H}_i)$, $i = 1, 2$ be two nondegenerate representations of A. Then $\pi_1 \oplus \pi_2 : A \to B(\mathcal{H}_1 \oplus \mathcal{H}_2)$ is not irreducible. (Evidently we don't bother with calling things "reducible".)
(2) Notice that a character on a C*-algebra is a pure state. (Indeed, for any states ϕ_1, ϕ_2 and $\alpha_1, \alpha_2 \in (0, 1)$ with $\alpha_1 + \alpha_2 = 1$, the map $\alpha_1\phi_1 + \alpha_2\phi_2$ will not be multiplicative.) It turns out ([8, Lemma 1.9.10]) that you can use a Krein-Milman argument to strengthen parts (1) and (2) of Exercise 7.19 to hold for pure states/irreducible representations. Then an argument like part (3) will allow you to prove the conclusion of Corollary 7.13 where F consists of all pure states of A.
(3) Not every C*-algebra has a faithful irreducible representation. Those that do are called *primitive*.

Exercise 8.3. If $\pi : A \to B(\mathcal{H})$ is irreducible, what does that say about the von Neumann algebra $\pi(A)''$? (Looking for a one word answer.)

Exercise 8.4. If π is not irreducible, and V is a proper invariant subspace for π, use Theorem 1.16 from the Prerequisite Notes to show that $\mathcal{H} = V \oplus V^\perp$, where $V^\perp$ is also invariant for π.

Proposition 8.5. *If A is a* C*-*algebra which is finite dimensional as a vector space, then*

$$A \cong \bigoplus_{s=1}^{j} M_{n(s)}(\mathbb{C})$$

is a finite direct sum of matrix algebras.

Proof. Suppose that A is a finite dimensional C*-algebra. By GNS and Exercise 7.21, view A as a subalgebra of $B(\mathcal{H})$, where $\mathcal{H}$ is finite dimensional. Thus, A is an algebra of *compact* operators.

It turns out [8, Corollary I.10.6] that every irreducible representation of $K(\mathcal{H})$ is unitarily equivalent to the identity representation. Moreover, Exercise 8.4 enables us to decompose the (identity) representation of A on $\mathcal{H}$ into a direct sum of irreducible representations $\pi_i : A \to B(\mathcal{H}_i)$ where $\mathcal{H} = \bigoplus_i \mathcal{H}_i$. Then each $\mathcal{H}_i$ must be finite dimensional, and we must have only finitely many terms in this direct sum decomposition, since $\mathcal{H}$ is finite dimensional. In other words, if $\mathcal{H}_i \cong \mathbb{C}^{n_i}$, then $\pi_i(A) \cong M_{n_i}$. Thus,

$$A = \bigoplus_i \pi_i(A) \cong \bigoplus_i M_{n_i}$$

as desired. ∎

This might be a good time to look back at Appendix 6 from the Prerequisite Notes.

Definition 8.6. A C*-algebra A is an *AF algebra* or *approximately finite dimensional* C*-*algebra* if A is the inductive limit of a sequence of finite-dimensional C*-algebras.

One has to prove that inductive limits always exist in the category of C^*-algebras and $*$-homomorphisms; cf. [11, Theorem 6.1.2 and preceding paragraphs].

The following Proposition was mentioned in the Prerequisite Notes, but not proved there.

Proposition 8.7. *If $A = \overline{\bigcup_n A_n}$ is the norm closure of an increasing union of subalgebras $A_n \subseteq A_{n+1} \subseteq \cdots \subseteq A$, then A is the inductive limit of the directed system (A_n, ι_{mn}) where $\iota_{mn} : A_n \to A_m$ is the inclusion map.*

Proof. It suffices to check that A satisfies the universal property of the inductive limit. So, suppose that B is a C*-algebra and that we have $*$-homomorphisms $\psi_n : A_n \to B$ such that $\psi_m \circ \iota_{mn} = \psi_n$ whenever $n \leq m$. Given $a \in A$, write $a = \lim_{n\to\infty} a_n$ where $a_n \in A_n$. The fact that our connecting maps are inclusions means that if $m \geq n$, $a_n = \iota_{mn}(a_n) \in A_m$. Thus, if N is large enough that $\|a_m - a_n\| < \varepsilon$ if $m \geq n \geq N$, then

$$\|\psi_m(\iota_{mn} a_n) - \psi_m(a_m)\| = \|\psi_m(a_n - a_m)\| < \varepsilon.$$

As $\psi_m \circ \iota_{mn} = \psi_n$, it follows that $(\psi_n(a_n))_n$ is Cauchy in B. We define $\psi : A \to B$ by $\psi(a) = \lim_n \psi_n(a_n)$ if $a = \lim_n a_n$ with $a_n \in A_n$. □

Exercise 8.8. Complete the proof of Proposition 8.7 by showing that ψ is well-defined (independent of the choice of sequence $(a_n)_n$); $*$-preserving; and multiplicative.

Example 8.9 (cf. Example 6.3 from the Prerequisite Notes)**.** $K(\ell^2)$ is an AF algebra. To see this, write P_n for the projection onto $\text{span}\{e_1, \dots, e_n\}$ and observe that $\mathrm{M}_n \cong P_n K(\ell^2) P_n$. Since $\overline{\bigcup_n P_n K(\ell^2) P_n} = \overline{FR(\ell^2)} = K(\ell^2)$, the result follows by applying the previous Proposition.

Remark 8.10. In the above example, we were discussing the compact operators on a fixed $\mathcal{H} = \ell^2$. However, (cf. Exercise 8.37 from the Prereqs) if two Hilbert spaces $\mathcal{H}, \mathcal{K}$ have the same dimension, with orthonormal bases $\{\xi_n\}_n, \{\eta_n\}_n$ respectively, then the map $U : \mathcal{H} \to \mathcal{K}$ given by $U(\xi_n) = \eta_n$ is a unitary. In particular (this is another **exercise**) the map $\mathrm{Ad}(U) : B(\mathcal{H}) \to B(\mathcal{K})$ given by

$$\mathrm{Ad}(U)(T) = UTU^*$$

is a C*-algebra isomorphism. In particular, it takes $FR(\mathcal{H})$ to $FR(\mathcal{K})$ and $K(\mathcal{H})$ to $K(\mathcal{K})$.

So, if $\mathcal{H}$ is *any* Hilbert space with a countable orthonormal basis, then $K(\mathcal{H})$ is isomorphic to $K(\ell^2)$ (and in particular is an AF algebra). Because of this, and the fact that algebras of compact operators are (as we'll see) both ubiquitous and indispensable, we often talk about "the compact operators" as shorthand for $K(\ell^2)$, or $K(\mathcal{H})$ for any separable Hilbert space $\mathcal{H}$. In the literature, the Hilbert space is often dropped altogether, and the compact operators are denoted $\mathcal{K}$ (not to be confused with the Hilbert space $\mathcal{K}$ that we have occasionally used in these notes).

By construction, Example 6.3 of the Prerequisite Notes describes an AF algebra. Here it is again.

Example. Let $A_n = M_{2^n}(\mathbb{C})$ be the algebra of $2^n \times 2^n$ matrices with maps $\phi_{n,n+1} : M_{2^n}(\mathbb{C}) \to M_{2^{n+1}}(\mathbb{C})$ defined by

$$x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}.$$

Letting $\phi_{n,m} := \phi_{m,m-1} \circ \cdots \circ \phi_{n,n+1}$ whenever $m > n$, we see that by construction this forms a directed system. Since these are inclusions, one can identify the inductive limit with $\bigcup_{n\in\mathbb{N}} A_n$. ■

This is a particularly important one, known as M_{2^∞} or the CAR algebra. In fact, it's an example of a UHF algebra.

Definition 8.11. An AF algebra A is a *UHF* or *uniformly hyperfinite* algebra if A is the inductive limit of a sequence of full matrix algebras, where the connecting maps are unital embeddings.

Exercise 8.12. Is $K(\ell^2)$ a UHF algebra?

Example 8.13. [8, Example III.3.7] One can obtain quite different C*-algebras from the same sequence of finite-dimensional C*-algebras (A_n), if one uses different connecting maps.

For example, let $A_n = \mathbb{C}^{2^n}$. On the one hand, let X denote the standard middle-third Cantor set, so that $X = \bigcap_n C_n$, where $C_n \subseteq [0,1]$ is the collection of 2^n intervals that remain after step n in the construction of X. We can construct $C(X)$ as an inductive limit of the algebras A_n, by identifying A_n with the set of functions on C_n that are locally constant. That is, we identify $\vec{z} = (z_1, \ldots, z_{2^n}) \in \mathbb{C}^{2^n}$ with the locally constant function $f_{\vec{z}}$ on C_n which takes the value z_1 on the left-most interval of C_n, the value z_2 on the next interval, and so on.

In this case, since $C_n \supseteq C_{n+1}$, the connecting maps $\iota_n : A_n \to A_{n+1}$, and the structure maps $\phi^n : A_n \to C(X)$, are given by restriction. It follows that the connecting maps are injective, so $\varinjlim(A_n, \iota_n) = \overline{\bigcup_n A_n}$ by Proposition 8.7. And a straightforward $\varepsilon - \delta$ proof will show you that the set of functions which are constant on some C_n is dense in $C(X)$ – that is, $C(X) = \overline{\bigcup_n A_n} = \varinjlim(A_n, \iota_n)$.

Under the identification of $\mathbb{C}^{2^n}$ with the locally constant functions on C_n, what does the map $\iota_n : A_n \to A_{n+1}$ look like? By construction, the two left-most intervals of C_{n+1} both lie in the left-most interval of C_n; the third- and fourth-leftmost intervals of C_{n+1} lie in the second-leftmost interval of C_n; and so on. In other words, $\iota_n(z_1, \ldots, z_{2^n}) = (z_1, z_1, z_2, z_2, \ldots, z_{2^n}, z_{2^n})$.

On the other hand, consider the space $Y = \{0\} \cup \{1/n : n \in \mathbb{Z}_{>0}\}$. Write $B_n \subseteq C(Y)$ for the set of functions which are constant on $[0, 2^{-n}]$. Then

$$B_n \cong C(\{1/k : 1 \leq k \leq 2^n\}) \cong \mathbb{C}^{2^n} \cong A_n.$$

In B_n, $(z_1, \ldots, z_{2^n}) \in \mathbb{C}^{2^n}$ is identified with the function $g_{\vec{z}} \in B_n \subseteq C(Y)$ which satisfies

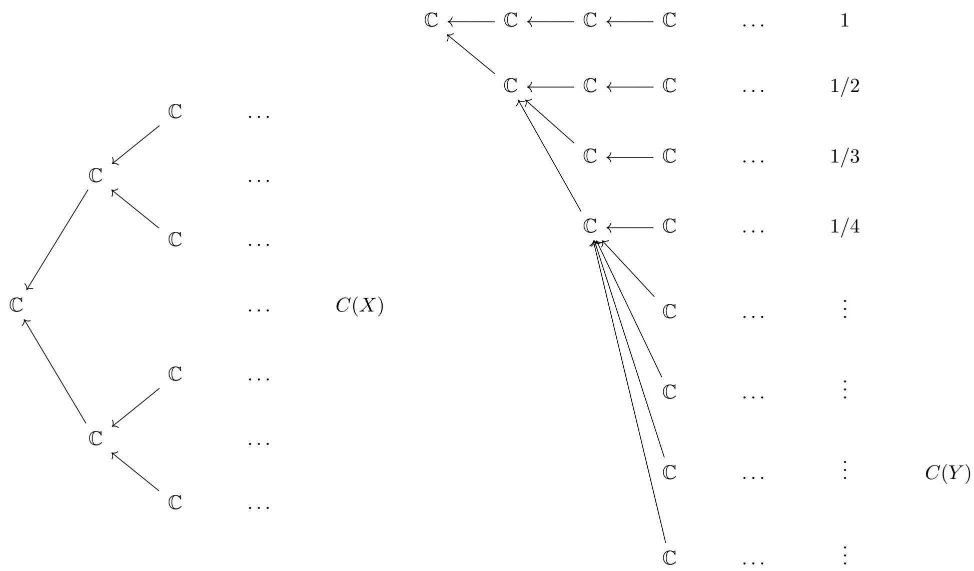
$$g_{\vec{z}}(y) = \begin{cases} z_k, & y = 1/k \text{ for } k \leq 2^n \\ z_{2^n}, & \text{else} \end{cases}$$

Again, the connecting maps $j_n : B_n \to B_{n+1}$ are given by inclusion, and $\bigcup_n B_n$ is dense in $C(Y)$, so $C(Y) = \varinjlim(B_n, j_n)$. But $C(Y) \not\cong C(X)$, since X and Y are not homeomorphic topological spaces.

What do the connecting maps j_n look like when we identify B_n with $\mathbb{C}^{2^n}$? Since $g_{\vec{z}} \in B_n$ takes the value z_{2^n} on $[0, 2^{-n}]$, its image in B_{n+1} is $g_{\vec{w}}$ where $\vec{w} = (z_1, z_2, \ldots, z_{2^n}, z_{2^n}, \ldots, z_{2^n})$ (with z_{2^n} occurring $2^n + 1$ times). That is, $j_n(z_1, \ldots, z_{2^n}) = (z_1, z_2, \ldots, z_{2^n}, z_{2^n}, \ldots, z_{2^n})$.

One sees the difference between these two AF algebras even more clearly via the *Bratteli diagram* of the AF algebras. If $A = \varinjlim(A_n, \phi_n)$, with $A_n = \bigoplus_{j=1}^{k(n)} M_{\ell(j)}$, and the connecting maps $\phi_n : A_n \to A_{n+1}$ are inclusions, the Bratteli diagram consists of $\mathbb{N}$ levels, with $k(n)$ nodes at each level, and an edge from a node v at level $n+1$ to a node w at level n if ϕ_n maps the wth matrix algebra into the vth matrix algebra.[16] For example, below are the Bratteli diagrams for $\varinjlim(A_n, \iota_n)$ and $\varinjlim(B_n, j_n)$.

[16] Here we see the disadvantage of our convention for graph algebras. The standard (and arguably more natural) convention for Bratteli diagrams has the edges pointing the other direction. But this convention makes for a neat connection between AF algebras and graph C*-algebras. See Theorem 8.19 below.



Exercise 8.14. Show that any AF algebra has an approximate identity which consists of an increasing sequence of projections.

Exercise 8.15. Show that any AF algebra is isomorphic to a direct limit of finite-dimensional C*-algebras with *injective* connecting maps.

Exercise 8.16. Show that if $A = \varinjlim(A_n, \phi_n)_n$ and we set $B_n := A_{n+k}$ for a fixed $k \geq 0$, then we also have $A = \varinjlim(B_n, \phi_{n+k})_n$.

One can have two different directed systems that give rise to the same C*-algebra. An example is the UHF algebra $M_{2^\infty 3^\infty} = \varinjlim(A_n, \iota_n) = \varinjlim(B_n, \iota_n)$, where

$$A_n = \begin{cases} M_{2^{n/2}3^{n/2}}, & n \text{ even} \\ M_{2^{(n+1)/2}3^{(n-1)/2}}, & n \text{ odd;} \end{cases} \qquad B_n = \begin{cases} M_{2^{n/2}3^{n/2}}, & n \text{ even} \\ M_{2^{(n-1)/2}3^{(n+1)/2}}, & n \text{ odd.} \end{cases}$$

The nodes at odd levels in the Bratteli diagrams of $\varinjlim(A_n, \iota_n)$ and $\varinjlim(B_n, \iota_n)$ are not isomorphic, nor is the number of edges between levels.

Fortunately, there is a complete invariant for AF algebras – a way to tell whether or not two AF algebras are isomorphic. G. Elliott proved in 1978 that the ordered K-theory $(K_0(A), K_0(A)_+, [1])$ of an AF algebra is a classifying invariant for A, in that given two AF algebras A, B, their K-theory groups are order isomorphic – $(K_0(A), K_0(A)_+, [1_A]) \cong (K_0(B), K_0(B)_+, [1_B])$ – if and only if $A \cong B$. You'll hear about K-theory from José Carrión on Wednesday, and [8, Chapter IV] has a proof of Elliott's classification theorem for AF algebras.

Because AF algebras are quite tractable, it's natural to ask which C*-algebras are subalgebras of AF algebras. That is, given a C*-algebra A, when can we find an injective $*$-homomorphism $\phi : A \to B$ for some AF algebra B? This simple-seeming question was only answered recently [Schafhauser 2018], under mild assumptions on A.

Exercise 8.17.

(1) Prove that $C([0, 1])$ is not an AF algebra.
(2) If X is the Cantor set, we've seen that $C(X)$ is AF. Show that a subalgebra of an AF algebra needn't be AF, by constructing an embedding of $C([0, 1])$ into $C(X)$.

However, despite the intricacy of the structure of the subalgebras of AF algebras, the lattice of ideals of an AF algebra is easy to describe: the ideals of an AF algebra are in bijection with hereditary saturated subsets of its Bratteli diagram [8, Theorem III.4.2]. The proof is the same as in the case of graph C*-algebras.

I hinted at the beginning of this section that if you view the Bratteli diagram of an AF algebra A as a graph E, then $C^*(E) \cong A$. This is not quite true, but it's almost true!

Given a (unital) AF algebra $A = \varinjlim(A_n, \phi_n)$, where $A_n = \bigoplus_{j=1}^{k(n)} M_{\ell(j)}$ and the maps ϕ_n are inclusions, let's assume that A_0 consists of a single summand and that the dimensions of the matrix algebras satisfy (for all j)

$$\ell(j) = \sum \{\ell(i) : i \leq k(n-1),\ \phi_{n-1}(M_{\ell(i)}) \subseteq M_{\ell(j)}\}.$$

That is, for any node in the Bratteli diagram, the arrows coming out of that node land in nodes whose dimensions collectively "fill up" the dimension of the node you started at.

Exercise 8.18. Convince yourself that this can always be achieved, by inserting levels between A_n and A_{n+1} if necessary.

Theorem 8.19. *Let A be a unital AF algebra as above, and let E be the associated Bratteli diagram. Let v be the unique vertex at level 0 of E. Then $A \cong p_v \mathrm{C}^*(E) p_v$.*

Proof. See [17, Proposition 2.12]. □

This theorem tells us that our original AF algebra is a *corner* of $\mathrm{C}^*(E)$. In fact, one can check **(Exercise)** that $p_v \mathrm{C}^*(E) p_v$ is a *full corner* of $\mathrm{C}^*(E)$, meaning the closed span of $\mathrm{C}^*(E) p_v \mathrm{C}^*(E)$ is all of $\mathrm{C}^*(E)$. This implies that A and $\mathrm{C}^*(E)$ are *Morita equivalent*, which means that in some sense they have the same category of representations. In particular, A and $\mathrm{C}^*(E)$ have the same K-theory. We will see more about Morita equivalence of two C^*-algebras in Section 14.

9. Crossed products

Suppose A is a unital C*-algebra. If $u \in A$ is a unitary, then the map $\phi_u : A \to A$ given by

$$\phi_u(a) = uau^*$$

is an *automorphism* of A. To see that ϕ_u is a $*$-homomorphism, note first that since $uu^* = u^*u = 1$, we have $\phi_u(ab) = \phi_u(a)\phi_u(b)$. It's also easy to check that $\phi_u(a^*) = \phi_u(a)^*$ for all $a \in A$. The fact that $\phi_{u^*}(\phi_u) = id = \phi_u(\phi_{u^*})$ means that ϕ_u is invertible.

However, not all automorphisms of a C*-algebra are given by conjugation by a unitary. For example, let $A = C(\mathbb{T})$. Because $C(\mathbb{T})$ is abelian, for any unitary $u \in C(\mathbb{T})$, we have $ufu^* = uu^*f = f$, so conjugation by a unitary does not give us any nontrivial automorphisms of $C(\mathbb{T})$. However, $C(\mathbb{T})$ does have nontrivial automorphisms! For example, fix $w \in \mathbb{T}$ and consider the automorphism α_w of A given by $\alpha_w(f)(z) = f(zw)$.

Exercise 9.1. Check that α_w is an automorphism.

We write $\mathrm{Aut}(A)$ for the group of automorphisms of a C*-algebra A.

Exercise 9.2. Check that $\mathrm{Aut}(A)$ is indeed a group. Do you need A to be unital to make sense of $\mathrm{Aut}(A)$?

Now, suppose G is a (discrete) group, A is a C*-algebra, and you have a group homomorphism $\alpha : G \to \mathrm{Aut}(A)$. From this data (sometimes called a *C*-dynamical system*), we can construct the *crossed product* $A \rtimes_\alpha G$ of A by (the action α of) G. Crossed products are another key family of examples of C*-algebras; the main questions revolve around how similar $A \rtimes_\alpha G$ is to A and to $C^*(G)$.

In these notes, we'll focus on crossed products by discrete groups; one can also define crossed products by locally compact Hausdorff groups, if you're willing to deal with integration against a Haar measure instead of finite sums.

Definition 9.3. Given a C*-dynamical system (A, G, α), a *covariant representation* is a unitary representation U of G on $\mathcal{H}$, together with a representation $\pi : A \to B(\mathcal{H})$, such that for all $a \in A$,

$$\pi(\alpha_g(a)) = U(g)\pi(a)U(g)^*.$$

We write $A \rtimes_{(\pi,U),\alpha} G := C^*(\{U_g, \pi(a) : g \in G, a \in A\})$.

That is, a covariant representation moves A to a bigger space where the automorphism α is indeed unitarily implemented.

Do we have any covariant representations? Yes!

Example 9.4. We know from GNS that we always have a faithful representation $\hat{\pi} : A \to B(\mathcal{H}_1)$ for some Hilbert space $\mathcal{H}_1$. Now, consider

$$\mathcal{H} := \ell^2(G, \mathcal{H}_1) = \{f : G \to \mathcal{H}_1 | \sum_{g \in G} \|f(g)\|^2 < \infty\}.$$

Define a unitary representation U of G on $\mathcal{H}$ by $U(g)f(h) = f(g^{-1}h)$, and define $\pi : A \to B(\mathcal{H})$ by

$$\pi(a)f(g) = \hat{\pi}(\alpha_g^{-1}(a))(f(g)).$$

Exercise 9.5. Convince yourself that U is indeed a unitary representation of G (so $U(g)^* = U(g^{-1})$) and that π is a $*$-representation of A.

Why did we pick this crazy formula? If you pay careful attention to what is a function and what is an operator and how things are defined, and use the fact that α is a group homomorphism, you can see that

$$\begin{aligned} U(g)\pi(a)U(g)^*f(h) &= U(g)\left(\pi(a)U(g^{-1})f\right)(h) = \left(\pi(a)U(g^{-1})f\right)(g^{-1}h) = \pi(a)\left(U(g^{-1})f\right)(g^{-1}h) \\ &= \hat{\pi}(\alpha_{h^{-1}g}(a))(U(g^{-1})f)(g^{-1}h) = \hat{\pi}(\alpha_{h^{-1}g}(a))(f(h)) \\ &= \hat{\pi}(\alpha_h^{-1}(\alpha_g(a)))(f(h)) \\ &= \pi(\alpha_g(a))f(h). \end{aligned}$$

So (π, U) is a covariant representation. That is, given *any* representation $\hat{\pi}$ of A on $\mathcal{H}_1$ and any C*-dynamical system (A, G, α), we get a covariant representation (π, U) of (A, G, α) on $\ell^2(G, \mathcal{H}_1)$. We call (π, U) the covariant representation *induced from* $\hat{\pi}$.

Exercise 9.6. Note that (convince yourself!) $\ell^2(G, \mathcal{H}_1) \cong \ell^2(G) \otimes \mathcal{H}_1$. Try rewriting the computations above with this perspective; it might actually be easier to follow, and it's a good warmup for when we get into tensor products of C*-algebras next week.

Another way to think about covariant representations of (A, G, α) is as representations of the following $*$-algebra, which is built out of A and G.

Definition 9.7. Given a C*-dynamical system (A, G, α), we define $A[G]$ (also sometimes $C_c(G, A)$ denoted) as follows:

$$C_c(G, A) = A[G] = \left\{ \sum_{g \in F} a_g u_g : F \subseteq G \text{ finite}, \ a_g \in A \right\}.$$

We make $C_c(G, A)$ into a $*$-algebra by

$$\begin{aligned} \left(\sum_{g\in F_1} a_g u_g\right)\left(\sum_{h \in F_2} b_h u_h\right) &:= \sum_{g \in F_1, h \in F_2} a_g (u_g b_h u_{g^{-1}}) u_{gh} \\ &= \sum_{g \in F_1, h\in F_2} a_g \alpha_g(b_h) u_{gh} = \sum_k \left(\sum_{h \in F_2} a_{kh^{-1}} \alpha_{kh^{-1}}(b_h)\right) u_k, \end{aligned}$$

and defining

$$\begin{aligned} \left(\sum_{g \in F} a_g u_g\right)^* &= \sum_{g\in F} u_g^* a_g^* = \sum_{g \in F} (u_{g^{-1}} a_g^* u_g) u_{g^{-1}} \\ &:= \sum_{g\in F} \alpha_{g^{-1}}(a_g^*) u_{g^{-1}} = \sum_{g : g^{-1} \in F} \alpha_g(a_{g^{-1}}^*) u_g. \end{aligned}$$

Notice that if A is unital, we can identify $G \subseteq C_c(G, A)$ via $g \mapsto u_g$; and we can always identify $A \subseteq C_c(G, A)$ via $a \mapsto a u_e$. The fact that $\alpha_e = id$ means that both of these embeddings respect all of the structure of the left-hand side; that is, $\{u_g\}_{g \in G}$ is a subgroup in $A[G]$ and $\{a u_e\}_{a \in A}$ is a $*$-subalgebra of $A[G]$.

Exercise 9.8. If $\rho : C_c(G, A) \to B(\mathcal{H})$ is a $*$-representation of $C_c(G, A)$, then $\rho|_A$ is a $*$-homomorphism and $\rho|_G$ is a unitary representation, and $(\rho|_A, \rho|_G)$ is a covariant representation of (A, G, α).

At the moment, we don't have a norm on $A[G]$; let's fix that. For any $*$-representation ρ of $A[G]$ as operators on a Hilbert space, we can define $A \rtimes_{\rho,\alpha} G$ to be the completion of $\rho(A[G])$ in the operator norm.

Exercise 9.9. Check that this definition is compatible with Definition 9.3: for any $*$-representation ρ of $A[G]$, we have $A \rtimes_{\rho,\alpha} G \cong A \rtimes_{(\rho|_A, \rho|_G), \alpha} G$.

As with group C*-algebras, we have a universal crossed product $A \rtimes_\alpha G$. To build it, we define a norm $\|\cdot\|_u$ on $A[G]$ by

$$\|x\|_u := \sup\{\|\rho(x)\| : \rho \text{ a } *\text{-representation of } A[G]\}.$$

The completion of $A[G]$ with respect to this norm is $A \rtimes_\alpha G$, the *universal crossed product* of A by (the action α of) G.

Exercise 9.10. How do we know that $\|\cdot\|_u$ is indeed a norm on $A[G]$? that is, how do we know that the only element of $A[G]$ with $\|x\|_u = 0$ is $x = 0$? *Hint:* Use Example 9.4.

Exercise 9.11. Let (A, G, α) be a C*-dynamical system. Prove that for any $*$-representation ρ of $A[G]$, we get a surjective $*$-homomorphism $\tilde{\rho} : A \rtimes_\alpha G \to A \rtimes_{\rho,\alpha} G$.

Remark 9.12. Some of you may have noticed that we have now described two seemingly different flavors of "universal" C*-algebras. On the one hand, we have C*(G) and $A \rtimes_\alpha G$, which were defined as completions of $\mathbb{C}[G]$ and $A[G]$ respectively with respect to the universal norm. On the other hand, we have the graph C*-algebras, which were defined using a universal property.

Good news: These definitions are in fact consistent. Proposition 5.10 and Exercise 5.17 combine to tell us that $C^*(G)$ satisfies the same sort of universal property that graph C*-algebras do. Namely,

$$C^*(G) = C^*(\langle \{u_g\}_{g\in G} : u_g^* = u_{g^{-1}}, u_{gh} = u_g u_h \rangle)$$

is generated by a family of unitaries satisfying certain algebraic relations (dictated by the group) and is universal in the sense that if $B = C^*_\rho(G)$ is any other C*-algebra generated by unitaries with the same relations, we have a surjective $*$-homomorphism $C^*(G) \to C^*_\rho(G)$ taking generators to generators.

The universal crossed product satisfies the same universal property:

$$A \rtimes_\alpha G = C^*(\langle \{a, u_g\}_{a\in A, g\in G} : u_g^* = u_{g^{-1}}, u_{gh} = u_g u_h, u_g a u_g^* = \alpha_g(a) \rangle).$$

That is, if we have any family of unitaries $\{U_g\}_{g\in G} \subseteq B$ which satisfy $U_g U_h = U_{gh}$ and $U_g^* = U_{g^{-1}}$, and any family $\mathcal{F}$ of elements in B that satisfy all the same relations as A (i.e., $\mathcal{F} = \pi(A)$ for a $*$-homomorphism $\pi : A \to B$) and such that for all $\pi(a) \in \mathcal{F}$ we have $u\pi(a)u^* = \pi(\alpha_g(a))$, then by Exercise 9.11 we have a surjective $*$-homomorphism

$$\rho : A \rtimes_\alpha G \to C^*(\langle \{\pi(a), U_g\}_{g\in G, a\in A}\}\rangle)$$

which takes a to $\pi(a)$ and u_g to U_g.

Let's look at some examples of crossed products.

Example 9.13. Let $A = C(\mathbb{T})$ and let $G = \mathbb{Z}$. Fix $w = e^{2\pi i\theta}$ in $\mathbb{T}$, and define $\alpha(f)(z) = f(wz)$. The universal crossed product $C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}$ is also known as the *rotation algebra* associated to $\theta \in (0, 1)$.

Recall that $C(\mathbb{T})$ is generated by the unitary $v(z) = z$. Moreover, in any covariant representation (π, U) of $(C(\mathbb{T}), \mathbb{Z}, \alpha)$ we will have a unitary $u := U(1)$, which generates the unitary representation of $\mathbb{Z}$, and

$$uvu^* = \alpha(v) = e^{2\pi i\theta} v.$$

That is, the universal property of $C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}$ tells us that this crossed product is the universal C*-algebra generated by two unitaries u, v such that

$$uv = e^{2\pi i\theta} vu.$$

This universal C*-algebra is also known as A_θ.

The structure of the rotation algebras was quite a hot topic about 40 years ago. It was eventually proved that all of the rotation algebras have the same K-theory, $K_0(A_\theta) = \mathbb{Z}^2$ for all θ, but that $A_\theta \cong A_{\tilde{\theta}}$ iff $\theta = 1 - \tilde{\theta}$.

Since $C(\mathbb{T}) = C^*(\{v\})$, the Gelfand-Naimark theorem tells us that $\sigma(v) = \mathbb{T}$. Moreover, since v is a unitary, and hence invertible,

$$\begin{aligned}
\lambda \in \sigma(u) &\iff u - \lambda I \text{ is not invertible} \iff v^*(u - \lambda I)v \text{ is not invertible} \\
&\iff v^*uv - \lambda I \text{ is not invertible} \iff e^{2\pi i\theta} u - \lambda I \text{ is not invertible} \\
&\iff w^{-1}\lambda \in \sigma(u).
\end{aligned}$$

In other words, $\sigma(u) = \mathbb{T}$ also.

The following result helps to explain the use of the real number θ, rather than the complex number $w = e^{2\pi i\theta}$, in the notation for A_θ.

Theorem 9.14. *The rotation algebra A_θ is simple iff $\theta \notin \mathbb{Q}$.*

Proof. If θ is rational, say $\theta = p/q$, consider the following elements of $M_q(\mathbb{C})$:

$$U = \begin{pmatrix} 1 & & & & \\ & w^{-1} & & & \\ & & w^{-2} & & \\ & & & \ddots & \\ & & & & w^{1-q} \end{pmatrix} \qquad V = \begin{pmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & & \ddots & \\ & & & 1 & 0 \end{pmatrix}.$$

Then one can compute that U, V are unitaries and that $UVU^* = wV$, using the fact that $w = e^{2\pi ip/q}$ satisfies $w^q = 1$. The universal property of A_θ therefore implies that there is a surjective $*$-homomorphism $\pi : A_\theta \to C^*(\{U, V\}) \subseteq M_q(\mathbb{C})$ with $\pi(u) = U$ and $\pi(v) = V$. However,

$$\sigma(U) = \{\lambda : (\lambda - 1)(\lambda - w^{-1})(\lambda - w^{-2}) \cdots (\lambda - w^{1-q}) = 0\} = \{1, w^{-1}, w^{-2}, \ldots, w^{1-q}\} \neq \mathbb{T}.$$

That is, $C^*(\{U,V\}) \neq A_\theta$, so $\ker\pi \neq 0$ is a nontrivial ideal of A_θ.

To see that A_θ is simple when $\theta \notin \mathbb{Q}$ will take a lot more work than we have time to do in lecture, so I will punt and just send you to [8, Theorem VI.1.4]. □

In fact, we have a more general theorem. Whenever X is a compact Hausdorff space and $\tilde{\alpha}$ is a homeomorphism of X, we obtain an action α of $\mathbb{Z}$ on $C(X)$ by pre-composing:

$$\alpha(f)(x) := f(\tilde{\alpha}^{-1}(x)).$$

It turns out (see [8, Theorem VIII.3.9]) that $C(X) \rtimes_\alpha \mathbb{Z}$ is simple iff X has no proper closed $\tilde{\alpha}$-invariant subsets. Equivalently, for any point $x \in X$, its orbit under $\tilde{\alpha}$,

$$\{\tilde{\alpha}^n(x) : n \in \mathbb{Z}\},$$

is dense in X. Such homeomorphisms are called *minimal*.

Example 9.15. Suppose $G = H \rtimes_\varphi K$ is a semidirect product of groups. That is, $\varphi : K \to \mathrm{Aut}(H)$ is a group homomorphism, and $G = \{(h,k)\}_{h\in H, k\in K}$, where the multiplication is given by $(h_1,k_1)(h_2,k_2) = (h_1\varphi_{k_1}(h_2), k_1k_2)$. Then $C^*(G) \cong C^*(H) \rtimes_\alpha K$, where the action α is given by

$$\alpha_k(u_h) = u_{\varphi_k(h)}.$$

Proof. First, notice that on the dense subalgebra $\mathbb{C}[H] \subseteq C^*(H)$, α is norm-decreasing (in fact norm-preserving) and invertible because it takes (unitary) generators to generators. It follows that each α_k extends to a norm-decreasing invertible map $C^*(H) \to C^*(H)$. To check that each α_k is an automorphism, then, we just have to check that it's multiplicative and $*$-preserving.

To see that α_k is multiplicative, let's compute:

$$\alpha_k\left(\sum_{h\in H} c_h u_h\right)\alpha_k\left(\sum_{j\in H} d_j u_j\right) = \sum_{j,h\in H} c_h d_j u_{\varphi_k(h)} u_{\varphi_k(j)} = \sum_{j,h\in H} c_h d_j u_{\phi_k(hj)} = \sum_{\ell\in H}\left(\sum_{h\in H} c_h d_{h^{-1}\ell}\right) u_{\phi_k(\ell)}$$
$$= \alpha_k\left(\left(\sum_{h\in H} c_h u_h\right)\left(\sum_{j\in H} d_j u_j\right)\right).$$

To see that each α_k is $*$-preserving, we use the fact that each φ_k is a group homomorphism:

$$\alpha_k\left(\sum_h c_h u_h\right)^* = \sum_h \overline{c_h} u_{\phi_k(h)^{-1}}; \qquad \alpha_k\left(\sum_h \overline{c_h} u_{h^{-1}}\right) = \sum_h \overline{c_h} u_{\phi_k(h^{-1})} = \sum_h \overline{c_h} u_{\phi_k(h)^{-1}}$$

as desired.

We'll use the universal properties of the algebras in question to define $*$-homomorphisms $\pi : C^*(G) \to C^*(H)_\alpha \rtimes K$ and $\psi : C^*(H) \rtimes_\alpha K \to C^*(G)$. Then we will show that π, ψ are mutually inverse to each other.

Write $C^*(G) = C^*(\langle\{u_{(h,k)} : h \in H, k \in K\}\rangle)$ and $C^*(H) \rtimes_\alpha K = C^*(\langle\{u_h, u_k\} : h \in H, k \in K\rangle)$. Define $\pi(u_{(h,k)}) = u_h u_k$. We check that π is multiplicative (on the generators):

$$\pi(u_{(h_1,k_1)} u_{(h_2,k_2)}) = \pi(u_{(h_1\varphi_{k_1}(h_2), k_1k_2)}) = u_{h_1\varphi_{k_1}(h_2)} u_{k_1k_2} = u_{h_1} u_{\varphi_{k_1}(h_2)} u_{k_1} u_{k_2},$$

whereas the fact that in $C^*(H) \rtimes_\alpha K$ we have $u_k u_h u_k^* = \alpha_k(u_h) = u_{\phi_k(h)}$ implies that

$$\pi(u_{(h_1,k_1)})\pi(u_{h_2,k_2)}) = u_{h_1} u_{k_1} u_{h_2} u_{k_2} = u_{h_1}(u_{k_1} u_{h_2} u_{k_1}^*) u_{k_1} u_{k_2} = u_{h_1}\alpha_{k_1}(u_{h_2}) u_{k_1k_2} = u_{h_1} u_{\varphi_{k_1}(h_2)} u_{k_1k_2}.$$

Similar computations will check that π is $*$-preserving. In other words, we have a $*$-representation of $\mathbb{C}[G]$ inside $C^*(H) \rtimes_\alpha K$, so the universal property of $C^*(G)$ implies that π gives a $*$-homomorphism which surjects onto $C^*(\{u_h u_k\}_{h\in H, k\in K})$. This C^*-algebra is in fact all of $C^*(H) \rtimes_\alpha K$, because by taking $h = e$ or $k = e$ we see that the generators $\{u_h, u_k : h \in H, k \in K\}$ of the crossed product are all in the image of π. It follows that $\pi : C^*(G) \to C^*(H) \rtimes_\alpha K$ is onto.

Similarly, we will define $\psi : C^*(H) \rtimes_\alpha K \to C^*(G)$ on the generators by $\psi(u_h) = u_{(h,e)}, \psi(u_k) = u_{(e,k)}$. By observing that $(e,k)^{-1} = (e, k^{-1})$ in G, we see that

$$\psi(u_k)\psi(u_h)\psi(u_k)^* = u_{(e,k)} u_{(h,e)} u_{(e,k)}^* = u_{(\varphi_k(h),k)} u_{(e,k^{-1})} = u_{(\varphi_k(h),e)} = \psi(\alpha_k(u_h)).$$

That is, the universal property of $\mathrm{C}^*(H) \rtimes_\alpha K$ implies that our definition above of ψ yields a $*$-homomorphism from $\mathrm{C}^*(H) \rtimes_\alpha K$ onto $\mathrm{Im}\psi \subseteq \mathrm{C}^*(G)$. As $u_{(h,e)}u_{(e,k)} = u_{(h,k)}$, every generator of $\mathrm{C}^*(G)$ is in $\mathrm{Im}\psi$, so ψ is onto.

Using the fact that ψ, π are $*$-homomorphisms, one computes immediately that $\psi \circ \pi = id$ and $\pi \circ \psi = id$ on the generators (and hence on the entire C*-algebras). □

9.1. **Reduced crossed products.** As with group C*-algebras, there's another flavor of crossed product we like to single out.

Definition 9.16. Let (A, G, α) be a C*-dynamical system. The *reduced crossed product* $A \rtimes_{\alpha,r} G$ is the completion of $C_c(G, A) = A[G]$ in the norm

$$\|x\|_r := \sup\{\|\pi(x)\| : \pi \text{ is induced from a representation } \hat{\pi} \text{ of } A\}.$$

In fact, we don't really need to take the supremum here.

Proposition 9.17. [5, Proposition 4.1.5] *For every faithful representation $\hat{\pi}$ of A, the induced representation π of $C_c(G, A)$ satisfies $\|\pi(x)\| = \|x\|_r$ for all $x \in C_c(G, A)$.*

Recall from Example 9.15 that if $G = H \rtimes_\varphi K$ is a semidirect product of groups, $\mathrm{C}^*(G)$ can be viewed as a crossed product C*-algebra. In fact, the same is true for the reduced crossed products.

Proposition 9.18. *If $G = H \rtimes_\varphi K$, then $\mathrm{C}^*_r(G) \cong \mathrm{C}^*_r(H) \rtimes_{\alpha,r} K$, where $\alpha_k(u_h) = u_{\varphi_k(h)}$.*

Proof. Recall that $\mathrm{C}^*_r(G) \subseteq B(\ell^2(G))$ and $\mathrm{C}^*_r(H) \subseteq B(\ell^2(H))$. So, define $V : \ell^2(G) \to \ell^2(K, \ell^2(H)) = \ell^2(K) \otimes \ell^2(H)$ on the basis vectors by $V(\delta_{(h,k)}) = \delta_k \otimes \delta_{\varphi_k^{-1}(h)}$. I claim that V is a unitary that intertwines the left regular representation of G with $\rho_{\lambda_H,U}$. That is, if $\hat{\lambda}_G, \hat{\lambda}_H$ denote the left regular representations of G and H respectively, and $\rho_{\lambda_H,U}$ for the representation of $\mathrm{C}^*_r(H)[K]$ induced from $\hat{\lambda}_H$,

$$V(\hat{\lambda}_G \circ \psi)V^* = \rho_{\lambda_H,U} \tag{9.1}$$

where ψ is defined as in Example 9.15: $\psi(u_h u_k) = u_{(h,k)}$. Since V is a unitary, we therefore have an isomorphism between the image of $\hat{\lambda}_G$ and that of $\rho_{\lambda_H,U}$. That is,

$$\mathrm{C}^*_r(G) = \mathrm{C}^*(\{\hat{\lambda}_G\}) \cong \mathrm{C}^*(\{\rho_{\lambda_H,U}\}) = \mathrm{C}^*_r(H) \rtimes_{\alpha,r} K.$$

Unitarity of V follows from the fact that it takes basis vectors to basis vectors. To see that Equation (9.1) holds, let's compute in a simple case. Fix $j, k \in K$ and $h, g \in H$. Viewing $u_g \in \mathrm{C}^*_r(H)$, we have

$$\rho_{\lambda_H,U}(u_g u_j)(\delta_h \otimes \delta_k) = (\lambda_H(u_g)U(j))(\delta_h \otimes \delta_k) = \hat{\lambda}_H(\alpha_{jk}^{-1}(u_g))\delta_h \otimes \delta_{jk} = \delta_{\varphi_{jk}^{-1}(g)h} \otimes \delta_{jk}.$$

On the other hand,

$$V(\hat{\lambda}_G(\psi(u_g u_j)))V^*(\delta_h \otimes \delta_k) = V\hat{\lambda}_G(u_{(g,j)})\delta_{(\varphi_k(h),k)} = V\delta_{(g\varphi_{jk}(h),jk)} = \delta_{\varphi_{jk}^{-1}(g)h} \otimes \delta_{jk}.$$ □

Another really important example of a reduced crossed product is the *uniform Roe algebra* of a discrete group G.

Example 9.19. Let G be a discrete group and consider $\ell^\infty(G)$ as the (abelian) subalgebra of $B(\ell^2(G))$ given by the diagonal matrices. The *uniform Roe algebra* $\mathrm{C}^*_u(G) = \ell^\infty(G) \rtimes_{\lambda,r} G$ is the reduced crossed product of $\ell^\infty(G)$ by the left-translation action of G.

You can also view $\mathrm{C}^*_u(G) \cong \mathrm{C}^*(\{\ell^\infty, \mathrm{C}^*_r(G)\}) \subseteq B(\ell^2(G))$ as the $*$-subalgebra of $B(\ell^2(G))$ generated by $\ell^\infty(G)$ and $\mathrm{C}^*_r(G)$, or as the C*-algebra generated by the operators T in $B(\ell^2(G))$ of *bounded propagation.* Check out [5, Section 5.1] for proofs of these statements.

Exercise 9.20. Since $\ell^\infty(G)$ is unital abelian, as a C*-algebra it's isomorphic to $C(X)$ for some compact Hausdorff space X. What is X? (How does it relate to G?)

10. Completely Positive Maps

This section gives a very quick introduction to completely positive maps for C*-algebraists. If this is your first time seeing such maps defined, we recommend ignoring the non-unital generalities for this go around. Once you have a grasp of the unital setting, you'll understand what's going on, and you will know where to look if you ever need the non-unital generalizations in the future. With the exception of a few examples, we will stick with the unital assumption in lecture.

We will focus mostly on understanding key examples of completely positive maps (Examples 10.7, 10.10, 10.11 and Exercise 10.13), the characterization of completely positive maps afforded by Stinespring's Dilation Theorem (Theorem 10.23), and an understanding Arveson's Extension Theorem (10.31) for completely positive maps into $B(\mathcal{H})$.

We give an overview of the proof of Stinespring's Dilation Theorem, which is a direct generalization of the GNS construction. In which case, it will be beneficial to have the GNS construction proof handy. This proof goes through some algebraic tensor products for vector spaces.

Section 10.1 establishes some preliminary results and delves into dilation techniques. We encourage you to read through the various dilation tricks and try the corresponding exercises in Section 10.1. These are valuable tools which we will likely not address in lecture.

We will have to forego several important facts and results on (completely) positive maps. For a full treatment, we recommend Vern Paulsen's book: [14, Chapters 2,3,6,7] as well as [5, Chapter 1].

This section concerns linear maps that preserve positivity even after matrix amplification. Before we get into the "what", let's briefly consider the "why". Basically, as great as *-hmomorphisms are, they aren't quite as abundant as we would like (especially if one is dealing with, say, a simple C*-algebra). So, we must content ourselves sometimes with linear *-preserving maps. But not just any linear *-preserving map will do; we need them to behave somewhat functorialy (read play nicely with usual C*-constructions like tensor products). Fortunately, so much of the nice structure of C*-algebras is wrapped up in their positive elements. Hence, we'll want linear maps that preserve positive elements. For technical reasons we shall see later in the tensor product section, we actually want them to preserve positive elements even after matrix amplification. And here our story begins...

Ignoring the norm for a moment, given a $*$-algebra A and $n \in \mathbb{N}$, define $M_n(A)$ to be the $n \times n$ matrices with entries in A (just as we would in more general ring theory):

$$\mathrm{M}_n(A) := \{[a_{ij}]_{i,j=1}^n \ : \ a_{ij} \in A, 1 \leq i, j \leq n\} \tag{10.1}$$

We will usually suppress the limits for the subscripts of an element $[a_{ij}]_{i,j=1}^n$ in $M_n(A)$, and instead write $[a_{ij}]$. When we mean to refer to a single coefficient in this matrix, we will write a_{ij} without the square brackets.

The vector space $M_n(A)$ is an algebra with matrix multiplication, and has an involution given by $[a_{ij}]^* = [a_{ji}^*]$ for all $[a_{ij}] \in \mathrm{M}_n(A)$.

Definition 10.1. Given a linear map $\phi : A \to B$ between $*$-algebras and $n \in \mathbb{N}$, define $\phi^{(n)} : M_n(A) \to \mathrm{M}_n(B)$ by

$$\phi^{(n)}([a_{ij}]) = [\phi(a_{ij})].$$

The map $\phi^{(n)}$ is often called a *matrix amplification* of ϕ.

Exercise 10.2. Show that every matrix amplification of a *-homomorphism is a *-homomorphism.

When A is a C*-algebra, there is a natural C*-norm on $\mathrm{M}_n(A)$, which is inherited from the norm on A. Recall from Exercise 4.25 from the Prerequisite Notes that $\mathrm{M}_n(B(\mathcal{H})) = B(\mathcal{H}^n)$ for any Hilbert space $\mathcal{H}$. Now (using Theorem 7.1), we faithfully represent A on some Hilbert space $\mathcal{H}$ with an injective $*$-homomorphism $\pi : A \to B(\mathcal{H})$. This induces a $*$-homomorphism $\pi^{(n)} : \mathrm{M}_n(A) \to \mathrm{M}_n(B(\mathcal{H})) = B(\mathcal{H}^n)$, which is also injective (check). Then we can define a norm on $\mathrm{M}_n(A)$ by $\|[a_{ij}]\| := \|\pi^{(n)}([a_{ij}])\|$ (injectivity implies this is a norm and not just a semi-norm), which will satisfy the C*-identity (because $\pi^{(n)}$ is a $*$-homomorphism).

The following inequality is a useful fact whose proof is outlined in [18, Exercise 1.13].

Proposition 10.3. *For any* C^**-algebra* A*,* $n \in \mathbb{N}$*, and* $[a_{ij}] \in M_n(A)$*, we have*

$$\max_{i,j}\{\|a_{ij}\|\} \le \|[a_{ij}]\| \le \sum_{ij} \|a_{ij}\|.$$

In particular, if $([a_{ij}^{(k)}])_{k=1}^{\infty}$ *is a sequence in* $M_n(A)$ *and* $[b_{ij}] \in M_n(A)$*, then* $\lim_k([a_{ij}^{(k)}]) = [b_{ij}]$ *iff* $\lim_k a_{ij}^{(k)} = b_{ij}$ *for all* $1 \le i, j \le n$*.*

10.1. **Preliminary results on completely positive maps.** Unlike with the Gelfand-Naimark Theorem for commutative C*-algebras, we will not start from scratch here. However, results in this section are developed nicely in [14, Chapter 2]. The proofs therein are well-written and easy to follow, but we are after bigger fish and therefore will just take these as means to an end.

Definition 10.4. A linear map $\phi : A \to B$ between C*-algebras is *positive* if it maps positive elements to positive elements, is n*-positive* if $\phi^{(n)}$ is positive, and is *completely positive* (c.p. or cp) if it is n-positive for all $n \in \mathbb{N}$. A completely positive map that is unital is abbreviated *ucp*.

Remark 10.5. In these notes, we take all cp maps to be linear and thus will not restate that assumption.

Remark 10.6. Beware: $M_n(A_+) \neq M_n(A)_+$. Indeed, this equality does not even hold for $A = \mathbb{C}$; for example, $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ is not even self-adjoint! However, we can at least say that for any $a_1, ..., a_n \in A$, the diagonal matrix $\operatorname{diag}(a_1, ..., a_n)$ is positive in $M_n(A)$ iff $a_1, ..., a_n \in A_+$. Indeed,

$$\begin{bmatrix} a_1 & 0 & 0 & \dots \\ 0 & a_2 & 0 & \dots \\ & & \ddots & \\ 0 & 0 & \dots & a_n \end{bmatrix} \ge 0 \Longleftrightarrow \begin{bmatrix} a_1 & 0 & 0 & \dots \\ 0 & a_2 & 0 & \dots \\ & & \ddots & \\ 0 & 0 & \dots & a_n \end{bmatrix} = \left(\begin{bmatrix} a_1 & 0 & 0 & \dots \\ 0 & a_2 & 0 & \dots \\ & & \ddots & \\ 0 & 0 & \dots & a_n \end{bmatrix}^* \begin{bmatrix} a_1 & 0 & 0 & \dots \\ 0 & a_2 & 0 & \dots \\ & & \ddots & \\ 0 & 0 & \dots & a_n \end{bmatrix} \right)^{1/2}$$

$$= \begin{bmatrix} a_1^*a_1 & 0 & 0 & \dots \\ 0 & a_2^*a_2 & 0 & \dots \\ & & \ddots & \\ 0 & 0 & \dots & a_n^*a_n \end{bmatrix}^{1/2} \overset{(!)}{=} \begin{bmatrix} |a_1| & 0 & 0 & \dots \\ 0 & |a_2| & 0 & \dots \\ & & \ddots & \\ 0 & 0 & \dots & |a_n| \end{bmatrix}.$$

For the last equality: Since $\operatorname{diag}(|a_1|, ..., |a_n|)^2 = \operatorname{diag}(a_1^*a_1, ..., a_n^*a_n)$, uniqueness of the square root then tells us that $\operatorname{diag}(a_1^*a_1, ..., a_n^*a_n)^{1/2} = \operatorname{diag}(|a_1|, ..., |a_n|)$, and the above string of equalities tells us $a_i = |a_i|$ for $i = 1, ..., n$.

One important class of examples of cp maps is the collection of positive linear functionals on a C^*-algebra (such as the states used in the GNS representation theorem).

Example 10.7. For a unital C*-algebra A, a positive linear functional $\phi \in A^*$ is completely positive. Indeed, (for the unital case) note that $\phi^{(n)} : M_n(A) \to M_n(\mathbb{C})$, so we check positivity by checking for positive-definiteness. To that end, let $\xi \in \mathbb{C}^n$ and $[a_{ij}] \in M_n(A)$ positive. Then by Exercise 3.11,

$$\begin{bmatrix} \bar{\xi}_1 & \dots & \bar{\xi}_n \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} \xi_1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \xi_n & 0 & \dots & 0 \end{bmatrix} = \begin{bmatrix} \sum_{i,j=1}^n \overline{\xi_i}\xi_j a_{ij} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & & \dots \;\dots & 0 \end{bmatrix} \tag{10.2}$$

is positive in $M_n(A)$. Then $\sum_{i,j=1}^n \overline{\xi_i}\xi_j a_{ij}$ is positive in A (by Remark 10.6), which means its image under ϕ is positive by assumption. Then we compute

$$\langle \phi^{(n)}([a_{ij}])\xi, \xi\rangle = \langle [\phi(a_{ij})]\xi, \xi\rangle = \left\langle \begin{bmatrix} \sum_{j=1}^n \phi(a_{1j})\xi_j \\ \vdots \\ \sum_{j=1}^n \phi(a_{n,j})\xi_j \end{bmatrix}, \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \right\rangle$$

$$= \sum_{i,j=1}^n \overline{\xi_i}\xi_j \phi(a_{ij}) = \phi(\sum_{i,j=1}^n \overline{\xi_i}\xi_j a_{ij}) \ge 0$$

Exercise 10.8. Show that the composition of completely positive maps is completely positive.

Exercise 10.9. Let $\phi : A \to B$ be a positive map between C*-algebras. Show that ϕ is $*$-preserving, i.e. $\phi(a^*) = \phi(a)^*$ for all $a \in A$.

Example 10.10. Conclude from Exercise 10.2 that any $*$-homomorphism is completely positive.

Example 10.11. Let $\psi : A \to B$ be a cp map between C*-algebras and fix $b \in B$. Then the map $\phi(a) := b^*\psi(a)b$ is linear and positive by Exercise 3.11. For each $n \in \mathbb{N}$ and $[a_{ij}] \in \mathrm{M}_n(A)_+$, note

$$\phi^{(n)}([a_{ij}]) = \begin{bmatrix} b^*\phi(a_{11})b & \dots & b^*\phi(a_{1n})b \\ \vdots & \ddots & \vdots \\ b^*\phi(a_{n1})b & \dots & b^*\phi(a_{nn})b \end{bmatrix} = \begin{bmatrix} b^* & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & b^* \end{bmatrix} \begin{bmatrix} \phi(a_{11}) & \dots & \phi(a_{1n}) \\ \vdots & \ddots & \vdots \\ \phi(a_{n1}) & \dots & \phi(a_{nn}) \end{bmatrix} \begin{bmatrix} b & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & b \end{bmatrix}.$$

Hence, when $\|b\| \leq 1$, ϕ is cp. Moreover, ϕ is *contractive:* $\|\phi(a)\| \leq \|a\|$ for all a.

Example 10.12. When $b \in A_+$, the map $A \to \overline{bAb}$ given by $a \mapsto bab$ is completely positive. When b is a projection in A, this map is often called a *compression*.

Exercise 10.13. Now consider a more concrete setting of $B(\ell^2)$, and consider the rank-n projection P defined on the standard basis $(e_i)_{i\in\mathbb{N}}$ by $Pe_i = 1$ if $i \leq n$ and $Pe_i = 0$ if $i > n$. If we write an operator $A \in B(\ell^2)$ as a matrix array, what does its image under the completely positive map $A \mapsto PAP$ look like? (This is where the word "compression" comes from.)

Now, we identify $PB(\ell^2)P \cong B(P\ell^2) \cong \mathrm{M}_n(\mathbb{C})$ (like in Example 8.9). These are $*$-isomorphisms, which means their composition with the above compression by P gives a completely positive map $B(\ell^2) \to \mathrm{M}_n(\mathbb{C})$.

Example 10.14. One important class of completely positive maps are conditional expectations. A *conditional expectation* is a contractive linear projection $E : A \to B$ from a C*-algebra onto a C*-subalgebra $B \subset A$ such that $Eb = b$ for all $b \in B$. By a theorem of Tomiyama, any conditional expectation is automatically completely positive and contractive. In this exercise, we consider a class of these that we will use a few times in these notes.

Recall that a finite dimensional C*-algebra has the form $B = \oplus_{j=1}^m \mathrm{M}_{l_j}(\mathbb{C}) \subset \mathrm{M}_L(\mathbb{C})$ where $L = \sum l_j$. For each j, let P_j denote the projection onto the jth summand of B, and define $\rho_j : \mathrm{M}_L \to \mathrm{M}_{l_j}$ as the compression $E_j(\cdot) = P_j \cdot P_j$ (where we identify $\mathrm{M}_{l_j}(\mathbb{C})$ with its copy in $\mathrm{M}_L(\mathbb{C})$). Then $E : \mathrm{M}_L(\mathbb{C}) \to B$, given by $\sum_j E_j$, is a ucp map (**exercise** check). (Why do we automatically know E is unital?)

We record the following for future use. The proof is short, but we leave it for [14, Theorem 3.9].

Proposition 10.15. *For any* C*-*algebra A and any compact Hausdorff space X, any positive map $\phi : A \to C(X)$ is cp.*

Remark 10.16. The same holds for maps $\phi : C(X) \to A$ (cf. [14, Theorem 3.11]). This is a theorem of Stinespring (not to be confused with the Stinespring Dilation Theorem in the next subsection).

Dilation Tricks: Though our overarching goals for this section are Theorems 10.23 and 10.31, we would be doing a disservice to come this close to dilation tricks and not give you a feel for the techniques. Also, we'll want some of these facts later.

Lemma 10.17. *Let A be a unital* C*-*algebra and $a, b \in A$. Then $\|a\| \leq 1$ iff $\begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix}$ is positive in $M_2(A)$.*

Proof. Let $\pi : A \to B(\mathcal{H})$ be a faithful representation such that $\pi(1) = I$. Since π is faithful, a is positive if and only if $\pi(a)$ is positive. Recall an element in $\mathrm{M}_2(B(\mathcal{H}))$ is positive if and only if it is positive-definite. Let $a \in A$. If $\|a\| \leq 1$, then for any $\xi, \eta \in \mathcal{H}$, we have

$$\left\langle \begin{pmatrix} I & \pi(a) \\ \pi(a)^* & I \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right\rangle = \langle \xi, \xi\rangle + \langle \pi(a)\eta, \xi\rangle + \langle \xi, \pi(a)\eta\rangle + \langle \eta, \eta\rangle \geq \|\xi\|^2 - 2\|\pi(a)\|\|\eta\|\|\xi\| + \|\eta\|^2$$
$$\geq ||\xi||^2 - 2||\eta||||\xi|| + ||\eta||^2 = (||\xi|| - ||\eta||)^2 \geq 0.$$

On the other hand, if $\|a\| > 1$, then there exist unit vectors $\xi, \eta \in A$ such that $\langle a\eta, \xi\rangle < -1$, which would make the inner product above negative. $\square$

Definition 10.18. A linear map $\phi : A \to B$ between C*-algebras is *completely bounded* if

$$\sup_n \|\phi^{(n)}\| < \infty.$$

When ϕ is completely bounded (cb), define $||\phi||_{\text{cb}} := \sup_n ||\phi^{(n)}||$. When $||\phi||_{\text{cb}} \le 1$, ϕ is called *completely contrctive*.

Corollary 10.19. *Any completely positive map is completely bounded. Moreover, if A and B are unital* C*-*algebras and $\phi : A \to B$ is a completely positive map, then*

$$\|\phi(1)\| = \|\phi\| = ||\phi||_{cb}$$

We prove the case where ϕ is unital, i.e. $\phi(1) = 1$, which also means $\phi^{(n)}(1) = 1_{\mathrm{M}_n(A)}$ for all $n \in \mathbb{N}$. The more general case needs one additional fact and is addressed in [14, Proposition 3.6], but the main idea is exhibited in the proof of the unital case.

Proof. By definition, $\|\phi(1)\| \le \|\phi\| \le ||\phi||_{\text{cb}}$. Moreover, as ϕ (and thus $\phi^{(n)}$) is unital, $\|\phi^{(n)}(1)\| = \|1_{\mathrm{M}_n(B)}\| = 1$ for all $n \in \mathbb{N}$. Thus, it suffices to show $||\phi||_{\text{cb}} \le 1$. Fix $n \in \mathbb{N}$ and let $a = [a_{ij}] \in \mathrm{M}_n(A)$ with $\|a\| \le 1$ be arbitrary. Then by Lemma 10.17,

$$\begin{pmatrix} 1_{\mathrm{M}_n(A)} & a \\ a^* & 1_{\mathrm{M}_n(A)} \end{pmatrix} \in \mathrm{M}_{2n}(A)$$

is positive. Since ϕ is completely positive, $\phi^{(n)}$ is 2-positive, and so

$$\phi^{(2n)}\left(\begin{pmatrix} 1_{\mathrm{M}_n(A)} & a \\ a^* & 1_{\mathrm{M}_n(A)} \end{pmatrix}\right) = \begin{pmatrix} 1_{\mathrm{M}_n(B)} & \phi^{(n)}(a) \\ \phi^{(n)}(a)^* & 1_{\mathrm{M}_n(B)} \end{pmatrix}$$

is positive. By Lemma 10.17, this implies $\|\phi^{(n)}(a)\| \le 1$. As $a \in \mathrm{M}_n(A)$ was arbitrary, $||\phi^{(n)}|| \le 1$, and as $n \in \mathbb{N}$ was arbitrary, we may conclude $||\phi^{(n)}|| \le 1$ for all $n \in \mathbb{N}$. Hence, $||\phi||_{\text{cb}} = 1$, as desired. ☐

Corollary 10.19 says that any contractive and cp (e.g. ucp) map is completely contractive.

Exercise 10.20. Let A be a unital C*-algebra and $a \in A$ such that $\|a\| \le 1$. Show that the following is a unitary in $\mathrm{M}_2(A)$:

$$\begin{pmatrix} a & (1-aa^*)^{1/2} \\ (1-a^*a)^{1/2} & -a^* \end{pmatrix}.$$

The map $A \to \mathrm{M}_2(A)$ defined by sendign a to the above matrix is sometimes referred to as Halmos' Dilation.

Now that we've tried a few dilation tricks, we (you) are ready to give another proof of Proposition 7.27.

Exercise 10.21. Let $\pi : A \to B$ be a $*$-homomorphism between C*-algebras and $b \in \pi(A)$. Show that there exists $a \in A$ such that $\pi(a) = b$ which satisfies $\|a\| = \|b\|$.

(1) Consider the element $x = \begin{bmatrix} 0 & b \\ b^* & 0 \end{bmatrix} \in \mathrm{M}_2(B)$. Show that $\|x^*x\| = \|b^*b\|$.
(2) Apply Exercise 2.34 to x and $\pi^{(2)}$ to get some lift $y = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \in \mathrm{M}_2(A)$ (i.e. $\pi^{(2)}(y) = x$) with y self-adjoint and $\|y\| = \|x\| = \|b\|$.
(3) Show that y_{12} is a lift of b.
(4) Now use Proposition 10.3 to finish the argument. (Don't forget to mention why $\|y_{12}\| \le b$.)

We close by considering unitizations. Although the Hahn-Banach theorem guarantees that any bounded linear map on a closed subspace extends to a bounded linear map on a larger space, there may be no completely positive extension of a positive map. This does hold in certain settings, however, in particular with unitizations. The proof is short but digs into some surprisingly technical aspects of double duals of C*-algebras, so we leave it to you to read [5, Proposition 2.2.1].

Proposition 10.22. *Let A and B be* C*-*algebras with A non-unital and B unital, and let $\phi : A \to B$ be a cpc map. Then ϕ extends to a ucp map $\tilde{\phi} : \tilde{A} \to B$, which is given by*

$$\tilde{\phi}(a + \lambda 1_{\tilde{A}}) = \phi(a) + \lambda 1_B.$$

10.2. **Stinespring's Dilation Theorem.** We saw in the previous subsection that compressing a $*$-homomorphism yields a completely positive map. What Stinespring's Dilation Theorem tells us is that that's basically how *every* completely positive map arises! That's right—when we are working with completely positive maps, we are really just looking at "compressed" $*$-homomorphisms.[17] That's what makes Stinespring's theorem so powerful: cp (ucp) maps are more abundant than $*$-homomorphisms, but when you have a cp map, you can draw a lot of conclusions pertaining to its structure by appealing to its "Stinespring Dilation" $*$-homomorphism.

Theorem 10.23 (Stinespring's Dilation Theorem)**.** *Let A be a unital C^*-algebra and $\phi : A \to B(\mathcal{H})$ a cp map. Then there exists a Hilbert space $\mathcal{H}'$, a representation $\pi : A \to B(\mathcal{H}')$ and a linear map $V : \mathcal{H} \to \mathcal{H}'$ such that*

$$\phi(a) = V^*\pi(a)V$$

for every $a \in A$. In particular, $\|\phi\| = \|V\|^2 = \|V^*V\| = \|\phi(1)\| = ||\phi||_{cb}$. Moreover, if ϕ is unital, then V is an isometry and $V^* = P_{V\mathcal{H}}$ is the projection onto $V\mathcal{H} \subset \mathcal{H}'$. In this case we identify $\mathcal{H}$ with a subspace $V\mathcal{H} \subset \mathcal{H}'$ and have

$$\phi(a) = P_{\mathcal{H}}\pi(a)|_{\mathcal{H}}.$$

Remark 10.24. We have a few remarks on this.

(1) When ϕ is unital, we think of $\pi(a)$ as

$$\pi(a) = \begin{bmatrix} \phi(a) & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

where $T_{12} : \mathcal{H}^\perp \to \mathcal{H}$, $T_{21} : \mathcal{H} \to \mathcal{H}^\perp$ and $T_{22} : \mathcal{H}^\perp \to \mathcal{H}^\perp$ are some bounded linear maps. Notice how the unital case generalizes Example 10.13 (with $\pi = id$).

(2) There is a non-unital version. Follow [5, Remark 1.5.4].

(3) One usually hears the term "minimal Stinespring dilation." Consider a Stinespring representation $(\pi, \mathcal{H}', V)$ for $\phi : A \to B(\mathcal{H})$. Let $\mathcal{H}_0 \subset \mathcal{H}$ be the closed linear span of $\pi(A)V\mathcal{H}$, which is reducing for $\pi(A)$ (as in the von Neumann algebra notes), and hence the co-restriction $\pi : A \to B(\mathcal{H}_0)$ is a representation. Whenever $\pi(A)V\mathcal{H}$ is dense in $\mathcal{H}'$, the Stinespring dilation is unique up to unitary equivalence (see [14, Proposition 4.2].)

(4) If $\phi : A \to B$ is a cp map between C^*-algebras, we can use the Gelfand-Naimark theorem and just embed B into some $B(\mathcal{H})$ and use Stinespring there.

The proof is exactly a generalization of the GNS construction of a representation corresponding to a state. The technique in general is sometimes called "separation and completion": first you define a semi-norm (or semi-inner-product in this case), then you mod out by the null set to make it it a genuine norm (or inner product). Finally you complete the quotient space with respect to your new norm. Since we have already seen the technical side of the GNS proof, let's see the overarching idea this time around in order to better understand how to potentially use this technique in other settings. (For a proof that checks all the details, see [14, Theorem 4.1].)

Sketch of proof of Stinespring's Dilation Theorem. Let $\phi : A \to B(\mathcal{H})$ be a cp map, and consider the algebraic tensor product

$$A \odot \mathcal{H} := \text{Span}\{a \odot \xi \ : \ a \in A, \xi \in \mathcal{H}\}.$$

Define a sesquilinear form $\langle \cdot, \cdot \rangle : A \odot \mathcal{H} \to \mathbb{C}$ on elementary tensors $a \odot \xi, b \odot \eta \in A \odot \mathcal{H}$ by

$$\langle a \odot \xi, b \odot \eta \rangle = \langle \phi(b^*a)\xi, \eta \rangle_{\mathcal{H}},$$

and extend linearly to $A \odot \mathcal{H}$. Since ϕ is cp, this sesquilinear form will be positive semi-definite (i.e. $\langle x, x \rangle \geq 0$ for all $x \in A \odot \mathcal{H}$). It turns out the space consisting of such elements $\mathcal{N} = \{x \in A \odot \mathcal{H} \ : \ \langle x, x \rangle = 0\}$ is a closed subspace of $A \odot \mathcal{H}$, and thus we can take the quotient $(A \odot \mathcal{H})/\mathcal{N}$. The sesquilinear form $\langle \cdot, \cdot \rangle$ from before now induces a genuine inner product[18] $\langle \cdot, \cdot \rangle_{\mathcal{H}'}$ on $(A \odot \mathcal{H})/\mathcal{N}$ given by

$$\langle x + \mathcal{N}, y + \mathcal{N} \rangle_{\mathcal{H}'} := \langle x, y \rangle_{\mathcal{H}}.$$

[17] "Compressed" is in quotations because in the non-unital setting it will be conjugation but not necessarily by a projection as in Definition 4.2.1 in the von Neumann notes.

[18] Positive semidefinite forms satisfy Cauchy-Schwarz, and so it follows just as with GNS that $\mathcal{N} = \{x \in A \odot \mathcal{H} \mid \langle x, y \rangle, \ \forall\, y \in A \otimes \mathcal{H}\}$.

When we complete $(A \odot \mathcal{H})/\mathcal{N}$ with respect to the norm induced by $\langle \cdot, \cdot \rangle_{\mathcal{H}'}$, we get a Hilbert space, which we have already suggestively denoted by $\mathcal{H}'$.

Given $a \in A$, define $\pi(a) : A \odot \mathcal{H} \to A \odot \mathcal{H}$ by left multiplication: for $a \in A$ and $b \odot \xi \in A \odot \mathcal{H}$,

$$\pi(a)(b \odot \xi) = ab \odot \xi,$$

and extend linearly. A short computation shows that $\mathcal{N}$ is invariant under $\pi(a)$, and so $\pi(a)$ induces a linear map on the quotient $(A \odot \mathcal{H})/\mathcal{N}$, which we still denote by $\pi(a)$. Moreover, the same computation shows that $\|\pi(a)(x + \mathcal{N})\| \leq \|a\|\|x + \mathcal{N}\|$ for all $x + \mathcal{N} \in (A \odot \mathcal{H})/\mathcal{N}$ (where $\|x + \mathcal{N}\|^2 = \langle x + \mathcal{N}, x + \mathcal{N} \rangle$), so we can extend $\pi(a)$ to a bounded linear operator on all of $\mathcal{H}'$. One then checks that this is indeed a unital $*$-homomorphism.

Let 1 denote the unit of A, and define $V : \mathcal{H} \to \mathcal{H}'$ by $V(\xi) = (1 \odot \xi) + \mathcal{N}$. Then we compute for each unit vector $\xi \in \mathcal{H}$, using Exercise 1.57 from the Prerequisite Notes,

$$\|V\xi\|^2 = \langle 1 \odot \xi, 1 \odot \xi \rangle_{\mathcal{H}'} = \langle \phi(1^*1)\xi, \xi \rangle_{\mathcal{H}} \leq \|\phi(1)\|.$$

It follows that $\|V\| = \|\phi(1)\|$ and moreover that V is an isometry when ϕ is unital.

Finally, we conclude that for all $\xi, \eta \in \mathcal{H}$,

$$\langle V^*\pi(a)V\xi, \eta \rangle_{\mathcal{H}} = \langle \pi(a)V\xi, V\eta \rangle_{\mathcal{H}'} = \langle \pi(a)((1 \odot \xi) + \mathcal{N}), (1 \odot \eta) + \mathcal{N} \rangle_{\mathcal{H}'} = \langle \phi(a)\xi, \eta \rangle_{\mathcal{H}},$$

and hence $V^*\pi(a)V = \phi(a)$. □

Exercise 10.25. Describe in words how (the proof of) Stinespring's Dilation Theorem generalizes the Gelfand Naimark Segal Theorem. In particular, when ϕ is a state, what is $\mathcal{H}$? $A \odot \mathcal{H}$?

Remark 10.26. Yes, there is a generalization of Stinespring's Dilation Theorem called the Kasparov-Stinespring Dilation Theorem. This is phrased in either the language of Hilbert C*-modules (see [10] for a nice introduction) or multiplier algebras (in Kasparov's original paper). Stinespring's theorem admits several generalizations. For instance, there is one for maps that are just considered completely bounded, i.e. linear maps with $\sup_n \|\phi^{(n)}\| < \infty$. (This is the Paulsen-Wittstock Dilation Theorem.)

For the sake of seeing Stinespring's Dilation Theorem in action, we introduce another useful concept for ucp maps, called the *multiplicative domain*.

Definition 10.27. Let A and B be unital C*-algebras and $\phi : A \to B$ cpc. The set

$$\{a \in A : \phi(a)\phi(b) = \phi(ab) \text{ and } \phi(b)\phi(a) = \phi(ba)\ \forall b \in A\}$$

is a C*-subalgebra of A called the *multiplicative domain of* ϕ.

Notice that ϕ is a $*$-homomorphism when restricted to this set. In fact, this is the largest C*-subalgebra on which the ucp map acts as a $*$-homomorphism, though the fact that it is a C*-algebra requires proof. To prove this, we use Stinespring's Dilation theorem to prove the following alternative description.

Proposition 10.28. *Let A and B be unital* C*-*algebras and $\phi : A \to B$ cpc. Then*

$$\{a \in A : \phi(a)\phi(b) = \phi(ab) \text{ and } \phi(b)\phi(a) = \phi(ba)\ \forall\, b \in A\} = \{a \in A : \phi(a)^*\phi(a) = \phi(a^*a) \text{ and } \phi(a)\phi(a)^* = \phi(aa^*)\}.$$

Proof. Let A be a unital C*-algebra and $\phi : A \to B$ a cpc map. One inclusion is immediate. We will work through the other inclusion. Since B can be faithfully represented on some $B(\mathcal{H})$ (and the composition of that representation with ϕ is still cp), we assume $B \subset B(\mathcal{H})$ and view ϕ as a map into $B(\mathcal{H})$. Let $(\pi, V, \mathcal{H}')$ be a Stinespring Dilation for $\phi : A \to B(\mathcal{H})$, i.e. $\pi : A \to B(\mathcal{H}')$ is a representation of A and $V : \mathcal{H} \to \mathcal{H}'$ is a contraction such that $\phi(a) = V^*\pi(a)V$ for all $a \in A$. Then for any $a, b \in A$, we have

$$\begin{aligned}
\phi(ab) - \phi(a)\phi(b) &= V^*\pi(ab)V - V^*\pi(a)VV^*\pi(b)V \\
&= V^*\pi(a)1_{\mathcal{H}'}\pi(b)V - V^*\pi(a)VV^*\pi(b)V \\
&= V^*\pi(a)(1_{\mathcal{H}'} - VV^*)\pi(b)V
\end{aligned}$$

Now, suppose $a \in A$ so that $\phi(a^*a) = \phi(a)^*\phi(a)$ and $\phi(aa^*) = \phi(a)\phi(a)^*$. Since V is contractive, so is VV^*, and so by Exercise 3.11, $1_{\mathcal{H}'} - VV^*$ is a positive contraction, which has a unique positive square root. With that observation, we compute

$$\begin{aligned}0 &= \phi(a^*a) - \phi(a)^*\phi(a) = V^*\pi(a)^*(1_{\mathcal{H}'} - VV^*)\pi(a)V \\ &= V^*\pi(a)^*((1_{\mathcal{H}'} - VV^*)^{1/2})^2\pi(a)V \\ &= [(1_{\mathcal{H}'} - VV^*)^{1/2}\pi(a)V]^*[(1_{\mathcal{H}'} - VV^*)^{1/2}\pi(a)V].\end{aligned}$$

It follows (from say the C*-identity) that $(1_{\mathcal{H}'} - VV^*)^{1/2}\pi(a)V = 0$. With that, we let $b \in A$ and compute

$$\phi(ba) - \phi(b)\phi(a) = V^*\pi(b)(1_{\mathcal{H}'} - VV^*)\pi(a)V = 0.$$

A symmetric argument shows that $\phi(ab) = \phi(b)\phi(a)$ for all $b \in A$, which completes the argument. □

Exercise 10.29. Conclude that the multiplicative domain of a cpc map from a unital C*-algebra is a C*-subalgebra.

Exercise 10.30. Let A and B be C*-algebras and $\phi : A \to B$ a cpc map. Scan the proof above to find an argument showing the "Choi's Schwarz Inequality for completely positive maps": $\phi(a^*a) \geq \phi(a)^*\phi(a)$ for all $a \in A$.

10.3. **Arveson's Extension Theorem.** The other major theorem for completely positive maps (as far as C*-algebraists are usually concerned) is Arveson's Extension Theorem. Just as Stinespring's Dilation Theorem was a generalization of the GNS Construction Theorem, which was a generalization of the Gelfand-Naimark Theorem, Arveson's Extension Theorem is a generalization of Krein's Theorem, a strengthening of the Hahn-Banach Theorem for C*-algebras. On the other hand, where Stinespring's proof was a generalization of the proofs that came before, Arveson's proof builds on the proofs that came before.

Theorem 10.31 (Arveson's Extension Theorem)**.** *Let A be a unital* C**-algebra, $B \subset A$ a unital* C**-subalgebra, and $\phi : B \to B(\mathcal{H})$ a cp map. Then there exists a cp map $\tilde{\phi} : A \to B(\mathcal{H})$ extending ϕ, i.e. $\tilde{\phi}|_B = \phi$.*

Remark 10.32. In an abuse of categorical language[19], $B(\mathcal{H})$ is often called *injective* in the category of C*-algebras with morphisms given by cpc maps. A C*-algebra C is called *injective* if for any unital C*-algebras $B \subset A$, any ucp map $\phi : B \to C$ extends to a ucp map $\tilde{\phi} : A \to C$. Note that the Hahn-Banach theorem says that ϕ always extends to a *contractive linear map*, but we are not guaranteed a positive extension.

This theorem plays a big role in the next section when we see a characterization of nuclear C*-algebras in terms of completely positive maps. For now, we just give an idea of the proof via the results it generalizes.

Theorem 10.33 (Krein)**.** *Let A be a unital* C**-algebra, $B \subset A$ a unital* C**-subalgebra, and $\phi : B \to \mathbb{C}$ a positive linear map. Then ϕ extends to a positive map on A.*

Note that, because the codomain of ϕ and its extension is $\mathbb{C}$ and both maps are automatically completely positive by Exercise 10.7.

Next, one establishes for a C* algebra A a bijective correspondence

$$\left\{ \begin{array}{c} \text{cp maps} \\ A \to \mathrm{M}_n(\mathbb{C}) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{cp maps} \\ M_n(A) \to \mathbb{C} \end{array} \right\}$$

(as in [14, Theorem 6.2]) to get the following intermediate result:

Proposition 10.34. *Let A be a unital* C**-algebra, $n \in \mathbb{N}$, $B \subset A$ a unital* C**-subalgebra, and $\phi : B \to \mathrm{M}_n(\mathbb{C})$ completely positive. Then ϕ extends to a completely positive map $A \to \mathrm{M}_n(\mathbb{C})$.*

From this to Arveson's theorem, we take a completely positive map $\phi : A \to B(\mathcal{H})$ and an increasing net of finite rank projections $P_i \in B(\mathcal{H})$. Then each compression $\phi_i : A \to P_iB(\mathcal{H})P_i \cong \mathrm{M}_{\mathrm{rank}P_i}(\mathbb{C})$, given by $P_i\phi(\cdot)P_i$, is a completely positive map with completely positive extension. From here you take a point-ultraweak cluster point of the ϕ_i's (ask Brent and Rolando), and that's your cp extension of ϕ!

[19]It's an abuse of language because we always assume an embedding $B \subset A$ is a $*$-homomorphism embedding.

Exercise 10.35. Suppose $C \subset B(\mathcal{H})$ is a unital C*-subalgebra of $B(\mathcal{H})$ (meaning its unit is $1_{\mathcal{H}}$) and $E : B(\mathcal{H}) \to C$ is a conditional expectation (which we recall from Exercise 10.14 is completely positive by Tomiyama's theorem). Show that C is injective as in Remark 10.32.

Using Example 10.14, conclude that Arveson's Extension Theorem holds for all finite dimensional C*-algebras.

Remark 10.36. If you've peeked at some of the reference texts, you'll notice that many of the theorems from this section are given for operator systems. What are those? Notice that completely positive maps completely preserve the structure of positive elements in a C*-algebra. So, there is a lot to be gained from considering self-adjoint unital subspaces of C*-algebras. One way to define a (concrete) *operator system* is as a unital self-adjoint subspace of a C*-algebra. (Not necessarily norm-closed.) Arveson's extension theorem is usually stated with the hypothesis that $B \subset A$ is not a C*-algebra but an operator system inside A.

11. Completely Positive Approximations

This section introduces what is historically known as the "completely positive approximation property," which, in the hindsight provided by a major theorem of Choi-Effros and also Kirchberg (which we give next week), is now called nuclearity. In essence, a C*-algebra has the completely positive approximation property when it can be well approximated by cpc maps that factor through finite dimensional C*-algebras. This is, at its heart, a property of maps, which is where we start in section 11.1.

However, in the lecture, we will focus on nuclearity of C*-algebras (11.2) and hence will take the material in Section 11.1 for granted. We will go through the discussion on $K(\ell^2)$ at the beginning of this section in the context of the definition of nuclearity. We will prove Proposition 11.15 and Proposition 11.9 in the separable setting. Arveson's Extension Theorem will feature prominently.

Though we will not be able to treat it in lecture, we highly recommend reading the argument that commutative C*-algebras are nuclear (Proposition 11.10) and working out the hands-on example in Exercise 11.12.

Many of the C*-algebras we can get our hands on have some reasonable connection to finite-dimensional C*-algebras. AF algebras in particular were built out of finite-dimensional subalgebras. More generally, they can be approximated by their finite dimensional subalgebras in a way that can be generalized to a much larger class of C*-algebras. To get a better feeling for what we mean, let us start with a motivating example.

We know (Example 8.9) that $K(\ell^2)$ is built as a union of finite-dimensional algebras as follows:

$$K(\ell^2) = \overline{\bigcup_n P_n K(\ell^2) P_n}$$

where P_n is the projection onto $\operatorname{span}\{e_1, \ldots, e_n\}$. Since the projections $(P_n)_n$ form an approximate unit for $K(\ell^2)$, we have for each $T \in K(\ell^2)$,

$$\|T - P_n T P_n\| \to 0.$$

We saw in the previous section that the map $T \mapsto P_n T P_n$ is a completely positive contractive map. Compose that with the $*$-isomorphism $P_n K(\ell^2) P_n \cong \mathrm{M}_n(\mathbb{C})$, and we have a cpc map $\psi_n : K(\ell^2) \to \mathrm{M}_n(\mathbb{C})$. Moreover, when we compose that with the $*$-homomorphism embedding $\phi_n : \mathrm{M}_n \to P_n K(\ell^2) P_n \subset K(\ell^2)$, we can write

$$\|T - \phi_n \psi_n(T)\| \to 0.$$

This is called a *completely positive approximation* of $K(\ell^2)$, and the existence of such an approximation is what it means (in modern terms) to be nuclear.

For the sake of simplicity, many results here are not stated in their full generality. If you find this section interesting, we suggest [5, Chapter 2], which covers this material quite well, save a dearth of hands-on examples.

11.1. **Nuclear Maps.**

We start with the definition of a nuclear map between C*-algebras.

Definition 11.1. A cpc map $\theta : A \to B$ between C*-algebras is called *nuclear* if there exist cpc maps $\psi_i : A \to \mathrm{M}_{k(i)}(\mathbb{C})$ and $\phi_i : \mathrm{M}_{k(i)}(\mathbb{C}) \to B$, for $i \in I$, so that $\phi_i \circ \psi_i \to \theta$ in the point norm topology, i.e. for each $a \in A$,

$$\lim_{i \in I} \|\phi_i(\psi_i(a)) - \theta(a)\| = 0.$$

Remark 11.2. There's lots to say here. This idea is thoroughly researched and nuanced, and there are so many variations on the above definition. We'll keep these remarks brief.

- If A is separable, then it can be written as a countable union of finite subsets. Then we can choose the net I in Definition 11.1 to be a sequence.
- The requirements placed on the maps ψ_i and ϕ_i can vary. It turns out we could equivalently relax the contractive requirement. On the other hand, we could equivalently keep the requirement that they are cpc and demand moreover that they have certain (approximate) orthogonality preserving properties (known as order zero). There's a fair bit of research in this direction by Winter, Zacharias, Kirchberg, Hirshberg, Brown, and Carrion to name a few.

- The convergence in Definition 11.1 could have been given with respect the point-ultraweak (aka σ-weak) topology (in which case the map would be called *weakly nuclear*). This is the first step on the bridge between nuclearity for C*-algebras and semidiscreteness/hyperfiniteness for von Neumann algebras (ask Brent and Rolando what those terms mean), but we are getting ahead of ourselves.

This is really a local property, as the following proposition shows.

Proposition 11.3. *A cpc map $\theta : A \to B$ is nuclear iff for any $\varepsilon > 0$ and finite set $F \subset A$, there exists $n \in \mathbb{N}$ and cpc maps $\psi : A \to \mathrm{M}_n(\mathbb{C})$ and $\phi : \mathrm{M}_n(\mathbb{C}) \to B$ such that*

$$\|\phi(\psi(a)) - \theta(a)\| < \varepsilon$$

for each $a \in F$.

Proof. Suppose there exist cpc maps $\psi_i : A \to \mathrm{M}_{k(i)}(\mathbb{C})$ and $\phi_i : \mathrm{M}_{k(i)}(\mathbb{C}) \to B$ for $i \in I$ so that $\phi_i \circ \psi_i \to \theta$ in the point norm topology. Then for any $\varepsilon > 0$ and $F \subset A$ finite, we choose $i \in I$ so that $\|\phi_i(\psi_i(a)) - \theta(a)\| < \varepsilon$ for each $a \in F$.

Now, we assume the localized version. As in Example 5.5 from the Prerequisite materials we form a directed set

$$\{(\varepsilon, F) \;:\; \varepsilon > 0, F \subset A \text{ finite}\}.$$

For each (ε, F), let $\phi_{(\varepsilon,F)}$ be a cpc map so that $\|\phi_{(\varepsilon,F)}(\psi_{(\varepsilon,F)}(a)) - \theta(a)\| < \varepsilon$ for each $a \in F$. Then for each $a \in A$, we have the desired convergence. $\square$

Exercise 11.4. Show that a map $\theta : A \to B$ is nuclear if there exist finite dimensional C*-algebras F_i and cpc maps $\psi_i : A \to F_i$ and $\phi_i : F_i \to B$ so that $\phi_i \circ \psi_i$ converges pointwise in norm to θ.
Hint: Recall that a finite dimensional C*-algebra has the form $F = \oplus_{j=1}^m \mathrm{M}_{l_j}(\mathbb{C}) \subset \mathrm{M}_L(\mathbb{C})$ where $L = \sum l_j$, and use Example 10.14.

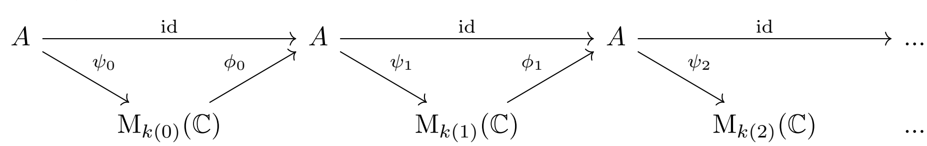
Exercise 11.5. Let A and B be C*-algebras and $C \subset B$ a C*-subalgebra. Show that if $\theta : A \to C$ is a nuclear map, then so is θ when viewed as a map from A to B. On the other hand, suppose we have a map $\rho : A \to C$ that is nuclear as a map from A to B. What could prevent ρ from being a nuclear map as a map from A to C?

11.2. **Completely Positive Approximation Property.**

Definition 11.6. A C*-algebra is *nuclear* if the identity map $\mathrm{id}_A : A \to A$ is nuclear, i.e. there exist cpc maps $A \xrightarrow{\psi_i} \mathrm{M}_{k(i)}(\mathbb{C}) \xrightarrow{\phi_i} A$ for $i \in I$ such that for each $a \in A$,

$$\|a - \phi_i(\psi_i(a))\| \to 0.$$

In the separable setting, the usual image one presents is something like the following approximately commutative diagram.



Remark 11.7. Sometimes these C*-algebras are called amenable. Sometimes for mathematical reasons—sometimes because the word "nuclear" in a grant application means one must fill out many many more forms.

A C*-algebra satisfying Definition 11.6 is also said to satisfy the *completely positive approximation property* (CPAP).

Example 11.8. It follows from Exercise 11.4 that finite dimensional C*-algebras are nuclear.

Proposition 11.9. *Ideals of nuclear* C*-*algebras are nuclear.*

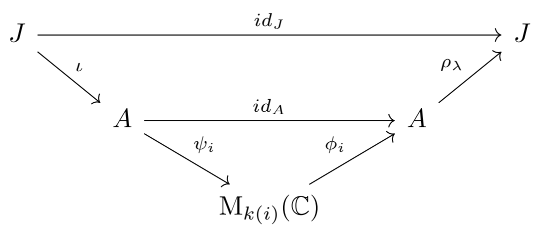
Proof. Suppose A is nuclear with completely positive approximation $A \xrightarrow{\psi_i} \mathrm{M}_{k(i)}(\mathbb{C}) \xrightarrow{\phi_i} A$ for $i \in I$. Let $J \triangleleft A$ be an ideal and $\{e_\lambda\}_\Lambda$ an approximate unit of J (with $0 \le e_\lambda \le e_\gamma \le 1$ when $\lambda \le \gamma$). Let $\iota : J \to A$ denote the inclusion of J into A (i.e. $\iota(a) = a$ for all $a \in J$). For each λ, define $\rho_\lambda : A \to J$ by $\rho_\lambda(a) = e_\lambda a e_\lambda$. Since each e_λ is self-adjoint and contractive, the maps ρ_λ are cpc by Exercise 10.11. Since the compositions of cpc maps are cpc (Exercise 10.8), for each i, λ, the maps $\psi'_{i,\lambda} := \psi_i \circ \iota : J \to \mathrm{M}_{k(i)}$ and $\phi'_{i,\lambda} := \rho_\lambda \circ \phi_i : \mathrm{M}_{k(i)} \to J$ are cpc. (Yes, the λ is a superflous index on ψ'_i.) Moreover, $\{(i,\lambda)\}_{I\times\Lambda}$ is a directed set with $(i,\lambda) \leq (j,\gamma)$ when $i \leq j$ and $\lambda \leq \gamma$.

Let $a \in J$ and $\varepsilon > 0$, and choose $(i_0, \lambda_0) \in I \times \lambda$ so that $\|a - \phi_i \circ \psi_i(a)\| < \varepsilon/2$ and $\|a - \rho_\lambda(a)\| < \varepsilon/2$ for each $i \geq i_0$ and $\lambda \geq \lambda_0$. Then for each $(i,\lambda) \geq (i_0, \lambda_0)$,

$$\begin{aligned}\|a - \phi'_{i,\lambda} \circ \psi'_{i,\lambda}(a)\| &= \|a - e_\lambda(\phi_i \circ \psi_i(a))e_\lambda\| \\ &\leq \|a - e_\lambda a e_\lambda\| + \|e_\lambda a e_\lambda - e_\lambda(\phi_i \circ \psi_i(a))e_\lambda\| \\ &\leq \|a - e_\lambda a e_\lambda\| + \|e_\lambda\|\|a - \phi_i \circ \psi_i(a)\|\|e_\lambda\| \\ &\leq \|a - e_\lambda a e_\lambda\| + \|a - \phi_i \circ \psi_i(a)\| \\ &< \varepsilon.\end{aligned}$$

□

In approximately commutative diagrams, the picture from the above proof looks like this.



It's not a proof, but it's a good intuition to guide the proof.

Proposition 11.10. *Abelian* C^**-algebras are nuclear.*

The proof uses what is known as a "partition of unity argument." Generalizing the idea of a partition of unity has proved very fruitful in certain areas of research in recent years, so we give this proof as an example.

We take for granted the fact from topology that, given any compact Hausdorff space X with open cover $U_1, ..., U_n$, there exist continuous functions $h_1, ..., h_n : X \to [0,1]$ so that $\mathrm{supp}(h_j) \subset U_j$ and $\sum_j h_j = 1$. (See [Theorem 2.13, Rudin, Real and Complex Analysis].) This is a *partition of unity* (in fact a rather nice one).

Proof. Let A be an abelian C^*-algebra. If A is not unital, then $A \lhd \tilde{A}$, and by Proposition 11.9, it suffices to show that $\tilde{A}$ is nuclear. So, we assume A is unital and moreover that $A = C(X)$ for some compact Hausdorff space X. Combining Proposition 11.3 and Exercise 11.4, we conclude that it suffices to show that for any $F \subset C(X)$ finite and $\varepsilon > 0$, there exists a finite dimensional C^*-algebra C (in our case, it will be $\mathbb{C}^n = \oplus_1^n \mathrm{M}_1(\mathbb{C})$) and cpc maps $C(X) \xrightarrow{\psi} C \xrightarrow{\phi} C(X)$ so that $\|f - \phi \circ \psi(f)\| < \varepsilon$ for every $f \in F$.

Let $F \subset C(X)$ be a finite subset and $\varepsilon > 0$. For each $x \in X$, let

$$U_x := \bigcap_{f \in F} f^{-1}(B_\varepsilon(f(x))).$$

Then $U_x \subset X$ is an open neighborhood of x such that for each $y \in U_x$ and $f \in F$, we have $|f(y) - f(x)| < \varepsilon$. Since X is compact, we choose $x_1, ..., x_n$ so that a finite subcover $U_{x_1}, ..., U_{x_n}$ covers X, and moreover for each $f \in F$ and $y \in U_i := U_{x_i}$,

$$|f(y) - f(x_i)| < \varepsilon.$$

Then we choose a partition of unity $h_1, ..., h_n : X \to [0,1]$ so that $\mathrm{supp}(h_j) \subset U_{x_j}$ and $\sum_j h_j = 1$.

Define $\psi : C(X) \to \mathbb{C}^n$ by $\psi(g) = (g(x_1), ..., g(x_n)) = \oplus_{j=1}^n ev_{x_j}$, where ev_{x_j} denotes the point evaluation $g \mapsto g(x_j)$. Then ψ is a unital $*$-homomorphism. Define $\phi : \mathbb{C}^n \to C(X)$ by

$$(\lambda_1, ..., \lambda_n) \mapsto \sum \lambda_i h_i.$$

Then ϕ is a positive map, which is moreover unital since $\phi(1, ..., 1) = \sum h_i = 1$. Hence by Proposition 10.15, it is ucp, and, in particular, cpc by Corollary 10.19.

So, we estimate for $f \in F$,

$$\begin{aligned}\|f - \phi \circ \psi(f)\| &= \| \left(\sum h_i\right) f - \sum f(x_i)h_i\| = \| \sum f h_i - f(x_i)h_i\| \\ &= \sup_{y\in X} |\sum (f(y) - f(x_i))h_i(y)| \le \sup_{y \in X} \sum |f(y) - f(x_i)| h_i(y) \\ &\le \sum \varepsilon h_i(y) = \varepsilon.\end{aligned}$$

□

Remark 11.11. There has been a significant push in the classification program for nuclear C*-algebras (that satisfy a nice list of adjectives) to come up with a non-commutative version of this partition of unity argument. With it comes certain non-commutative dimension theories (see for example Winter and Zacharias's paper on Nuclear Dimension).

Exercise 11.12. Partitions of unity are nicer when you have a concrete example. For each integer $n \ge 2$, cover $[0,1]$ by $2^n - 1$ open intervals of length $1/2^{n-1}$. (What are they? Also, we could start with $n = 1$, but it's too simple to help us pick up on a pattern.) Call this cover $\mathcal{U}_n$. Define (sketch) a partition of unity for $\mathcal{U}_n$. (Hint: think zig-zags.)

Now, construct a sequence of completely positive maps $C([0,1]) \xrightarrow{\psi_n} \mathbb{C}^{k_n} \xrightarrow{\phi_n} C([0,1])$, (what is k_n?) that give a completely positive approximation of $C([0,1])$.

Proposition 11.13. *Suppose for each finite subset $F \subset A$ and $\varepsilon > 0$, there exists a nuclear* C*-*subalgebra $B \subset A$ such that for each $a \in F$, there exists $b \in B$ such that $\|a - b\| < \varepsilon$. Then A is nuclear.*

Proof. By Proposition 11.3, it suffices to show that for any $\varepsilon > 0$ and finite set $F \subset A$, there exists $n \in \mathbb{N}$ and cpc maps $\psi : A \to \mathrm{M}_n(\mathbb{C})$ and $\phi : \mathrm{M}_n(\mathbb{C}) \to B$ such that

$$\|\phi(\psi(a)) - \theta(a)\| < \varepsilon$$

for each $a \in F$. Let $\{a_1, ..., a_m\} \subset A$ be a finite subset, $\varepsilon > 0$, and let $B \subset A$ nuclear so that for each a_j, there exists a $b_j \in B$ such that $\|a_j - b_j\| < \varepsilon/3$. Let $n \in \mathbb{N}$ and $\psi_B : B \to \mathrm{M}_n(\mathbb{C})$ and $\phi_B : B \to \mathrm{M}_n(\mathbb{C})$ be cpc maps so that $\|b_j - \phi_B\psi_B(b_j)\| < \varepsilon/3$ for each $1 \le j \le m$.

But how do we get a map ψ defined on all of A? Easy, since $\mathrm{M}_n(\mathbb{C}) = B(\mathbb{C}^n)$, the cpc map $\psi_B : B \to \mathrm{M}_n(\mathbb{C})$ extends to a cpc map $\psi : A \to \mathrm{M}_n(\mathbb{C})$ by Arveson's Extension Theorem.[20] Since $\phi_B : \mathrm{M}_n(\mathbb{C}) \to B \subset A$, we don't need to change it, so we choose $\phi = \phi_B$.

Now, all that's left is to compute for each $1 \le j \le m$:

$$\begin{aligned}\|a_j - \phi\psi(a_j)\| &\le \|a_j - b_j\| + \|b_j - \phi\psi(b_j)\| + \|\phi\psi(b_j) - \phi\psi(a_j)\| \\ &\le \|a_j - b_j\| + \|b_j - \phi\psi(b_j)\| + \|b_j - a_j\| \\ &< \varepsilon.\end{aligned}$$

□

Exercise 11.14. Using the above proposition, show that nuclearity is preserved under taking direct limits. Conclude that AF algebras are nuclear.

The above proof is perhaps a little abstract. Here's a version that's a little more tangible. First, we recall once more the construction of the CAR algebra:

Let $\mathrm{M}_{2^n}(\mathbb{C})$ be the algebra of $2^n \times 2^n$ matrices with maps $\phi_{n,n+1} : M_{2^n}(\mathbb{C}) \to M_{2^{n+1}}(\mathbb{C})$ defined by

$$x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}.$$

The inductive limit is denoted $\mathrm{M}_{2^\infty} = \overline{\bigcup_n \mathrm{M}_{2^n}(\mathbb{C})}$. Note that by construction, for each $n \in \mathbb{N}$, the copy of M_{2^n} in M_{2^∞} is unital.

Proposition 11.15. *The CAR algebra is nuclear.*

[20]Acutally, it's overkill here– one of the preliminary results leading up to Arveson's would work in finite dimensions.

Proof. For each $n \in \mathbb{N}_0$, define $\phi_n : \mathrm{M}_{2^n}(\mathbb{C}) \to \mathrm{M}_{2^\infty}$ be the inclusion where we identify M_{2^n} with its copy inside M_{2^∞}. The restriction of this map to its image is a $*$-isomorphism, so we call its inverse $\phi_n^{-1} : \phi_n(\mathrm{M}_{2^n}(\mathbb{C})) \to \mathrm{M}_{2^n}(\mathbb{C})$. This is a unital $*$-homomorphism from a C*-subalgebra of M_{2^∞} to $\mathrm{M}_{2^n}(\mathbb{C}) = B(\mathbb{C}^{2^n})$. So Arveson's Extension Theorem says ϕ_n^{-1} has a ucp extension $\psi_n : \mathrm{M}_{2^\infty} \to \mathrm{M}_{2^n}(\mathbb{C})$. So, we have ucp maps $\psi_n : \mathrm{M}_{2^\infty} \to \mathrm{M}_{2^n}(\mathbb{C})$ and $\phi_n : \mathrm{M}_{2^n}(\mathbb{C}) \to \mathrm{M}_{2^\infty}$. Moreover, for each $a \in \bigcup_n \mathrm{M}_{2^n}(\mathbb{C})$, there exists an $N \in \mathbb{N}$ so that $a \in \mathrm{M}_{2^n}(\mathbb{C})$ for all $n \geq N$, which means $\phi_n \circ \psi_n(a) = \phi_n \circ \phi_n^{-1}(a) = a$ for all $n \geq N$.

Now, suppose $a \in \mathrm{M}_{2^\infty}$, and $a_0 \in \bigcup_n \mathrm{M}_{2^n}(\mathbb{C})$ so that $\|a - a_0\| < \varepsilon/2$. Choose $N \in \mathbb{N}$ so that $\phi_n \circ \psi_n(a_0) = a_0$ for all $n \geq N$. Then for all $n \geq N$,

$$\|a - \phi_n \circ \psi_n(a)\| \leq \|a - a_0\| + \|a_0 - \phi_n \circ \psi_n(a_0)\| + \|\phi_n \circ \psi_n(a_0 - a)\| < \varepsilon. \qquad \square$$

Exercise 11.16. Generalize the proof of Proposition 11.15 to get another proof that all separable AF algebras are nuclear.
Hint: Consider a inductive (aka directed) system of finite dimensional C*-algebras (A_n, ι_{mn}) where $\iota_{mn} : A_n \to A_m$ is the inclusion map, and let A be the direct (inductive) limit of this system. Then use Exercise 10.35.

Chapter 2 in [5] does an excellent job of introducing the operations that do and do not preserve nuclearity. Since we do not wish to re-write their book, we will just collect them here. These range from easy exercises to deep theorems.

(1) Nuclearity passes to direct limits and direct sums ($\bigoplus_i A_i$) (but not direct products $\prod_i A_i$).
(2) Nuclearity passes to quotients.
There are essentially two proofs for this. The first is a consequence of Connes' Fields Medal work involving showing hyperfinite $\Leftrightarrow$ injective – ask Brent and Rolando. Otherwise, it follows from the fact that exactness (defined soon) passes to quotients. The proof of this (due to Kirchberg) is one of the most difficult proofs in C*-algebras, resting on some of the deepest and most difficult theorems in von Neumann algebra theory. See [5, Chapter 9] for a (not self-contained) outline.
(3) Nuclearity does not necessarily pass to subalgebras.
The easiest examples come from crossed products, which we'll see next week. (See [5, Remark 4.4.4].) For a more sophisticated appeal, we have Kirchberg's $\mathcal{O}_2$ embeddability theorem, which implies that the non-nuclear C*-algebra $\mathrm{C}^*_r(\mathbb{F}_2)$ embeds into the nuclear C*-algebra $\mathcal{O}_2$. (We will see next week why $\mathrm{C}^*_r(\mathbb{F}_2)$ is not nuclear. We take for granted that the Cuntz-Krieger algebras are nuclear.)
(4) Nuclearity passes to ideals (Proposition 11.9) (even hereditary subalgebras) and C*-subalgebras to which there exists a conditional expectation.
(5) Nuclearity passes to extensions, i.e. if $0 \to J \to A \to B \to 0$ is short exact and both J and B are nuclear, then so is A. (This one is easier with next week's characterization.)

We wrap up this section with a slight weakening of nuclearity that is still a very powerful property.

As we saw in Exercise 11.5, the range of a cpc map has a lot of bearing on whether or not it is nuclear. It may be that a C*-algebra fails to be nuclear but still has a faithful nuclear representation. These are still a nice class of C*-algebras.

Definition 11.17. A C*-algebra A is *exact* if there exists a faithful nuclear representation $\pi : A \to B(\mathcal{H})$.

Every nuclear C*-algebra is exact. Moreover, for nuclear C*-algebras, the map $\pi : A \to \pi(A)$ is nuclear. A non-nuclear example of an exact C*-algebra is $\mathrm{C}^*(\mathbb{F}_2)$ (due to Wasserman).

Exercise 11.18. Show that exactness *does* pass to C*-subalgebras. What does that tell you about every C*-subalgebra of a nuclear C*-algebra?

The name "exact" is hardly justified here. We will see it again later in the tensor product section, where it will make more sense.

12. Tensor Products of C*-algebras

Overall, sections 4.2 and 4.3 of the prerequisite notes will be treated as preliminary material in the lecture, which will focus more on sections 12.1 and 12.5. Section 12.2 goes into much more difficult problems concerning injectivity and exactness for tensor products. The point there is to just give a feel for the the questions and obstacles in both settings. With time, we will touch on the topics in lecture. Section 12.3 gives a tensor product characterization of nuclearity (Theorem 12.40) and highlights some important examples (Remark 12.43). We will mention these in lecture but without much discussion. Section 12.4 establishes an important class of examples (Theorem 12.47), which we will be sure to mention in lecture, but without much word on the proof.

Section 12.1 defines the two primarily studied C*-norms on tensor products. These are quite analogous to the universal and reduced norms for discrete groups, and we will explore several tensor product analogies to results we saw for groups, e.g. Corollary 12.18, Proposition 12.23, and Proposition 12.22. Section 12.5 justifies our use of *completely* positive maps. We will cover Example 12.50 and mention how Stinespring's Dilation theorem is used in the proof of Theorem 12.51.

The way you read these notes will depend on your background and comfort level. If algebraic tensor products are new to you, spend more time in section 4.2. Regardless of your comfort level with algebraic tensors, be sure you've digested Exercise 12.2, which is quite foundational to the later sections. If you are still shaky on Hilbert space operators, linger in section 4.3. If you feel comfortable with (assuming) the material in these sections, but still want some more fundamental examples and arguments under your belt, check out sections 12.4 and 12.3.

One of the most important constructions in C*-algebras is the tensor product. Given two C*-algebras A and B, we form a C*-tensor product $A\otimes_\alpha B$ by taking the $*$-algebraic tensor product $A\odot B$ and completing with some C*-norm. In this section, we consider the two most prominent ones. This section is taken heavily from the first half of [5, Chapter 3].

One word on notation. Because there is so much significance to the norm on a given tensor product, we will denote algebraic tensor products by $\odot$ and tensor products that are also complete with respect to a norm by $\otimes$ (possibly with decoration to denote which norm). Sometimes $\otimes$ is used in the literature to denote an algebraic tensor product, and sometimes it is used to indicate the normed tensor product space with the spatial tensor product norm Definition 12.11. Usually authors are good about warning you of this.

We are interested in particular in tensor products of C*-algebras. When A and B are C*-algebras, then the algebraic tensor product is a $*$-algebra with multiplication and involution defined on simple tensors as

$$(a\odot b)^* = a^*\odot b^* \quad \text{and} \quad (a_1\odot b_1)(a_2\odot b_2) = a_1a_2\odot b_1b_2,$$

and extended linearly to all of $A\odot B$. Recall from Section 10 where we defined a natural C*-norm on

$$\mathrm{M}_n(A) := \{[a_{ij}] \ : \ a_{i,j}\in A, 1\le i,j\le n\}. \tag{12.1}$$

Products of maps When we take the product of two $*$-homomorphisms $\psi_1 : A_1\to B$ and $\psi_2 : A_2\to B$, we are forced to impose an extra condition to guarantee that the product $\psi_1\times\psi_2$ is again a $*$-homomorphism: the images must commute, i.e. for each $a_1\in A_1$ and $a_2\in A_2$, $\psi_1(a_1)\psi_2(a_2) = \psi_2(a_2)\psi_1(a_1)$.

Exercise 12.1. Justify the claim above, i.e. the product $\psi_1\times\psi_2$ of two $*$-homomorphisms $\psi_1 : A_1\to B$ and $\psi_2 : A_2\to B$ is a $*$-homomorphism provided that the ranges $\psi_1(A_1)$ and $\psi_2(A_2)$ commute in B.

Exercise 12.2. Let A be any C*-algebra, $1\le n<\infty$, and let $E_{i,j}$ denote the matrix units on $\mathrm{M}_n(\mathbb{C})$ (i.e. the matrices with 1 in the i,j coordinate and 0 elsewhere). Define a map $\pi : \mathrm{M}_n(A)\to\mathrm{M}_n(\mathbb{C})\odot A$ by $\pi([a_{i,j}]) = \sum_{i,j=1}^n E_{i,j}\odot a_{ij}$. Show that this is an algebraic $*$-isomorphism.

We can also take the *tensor* product of two *-homomorphisms:

Proposition 12.3. *Suppose A_1A_2, B_1, B_2 are* C*-*algebras and $\phi_i : A_i\to B_i$, $i=1,2$ are *-homomorphisms. Then the* tensor product map

$$\phi_1\odot\phi_2 : A_1\odot A_2\to B_1\odot B_2$$

given by $\phi_1\odot\phi_2(a\odot b) = \phi_1(a)\odot\phi_2(b)$ for all $a\in A_1$, $b\in A_2$ is also a *-homomorphism.

Recall from Proposition 4.21 from the Prereqs that for any $a \in B(\mathcal{H}_1)$ and $b \in B(\mathcal{H}_2)$, there is a unique bounded operator $a \otimes b \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ given by

$$(a \otimes b)(\xi_1 \otimes \xi_2) = a\xi_1 \otimes b\xi_2, \ \forall\ \xi_1 \in \mathcal{H}_1, \xi_2 \in \mathcal{H}_2.$$

In infinite dimensions, we do not have $B(\mathcal{H}_1) \odot B(\mathcal{H}_2) = B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ (the former is no longer automatically closed). What we can say is that $B(\mathcal{H}_1) \cong B(\mathcal{H}_1) \otimes \mathbb{C}1_{\mathcal{H}_2}$ and $B(\mathcal{H}_2) \cong \mathbb{C}1_{\mathcal{H}_1} \otimes B(\mathcal{H}_2)$, and Proposition 4.21 from the Prereqs gives a natural *-homomorphism

$$B(\mathcal{H}_1) \odot B(\mathcal{H}_2) \to B(\mathcal{H}_1 \otimes \mathcal{H}_2).$$

From that, we get tensor products of representations.

Corollary 12.4. *Given two representations $\pi_i : A_i \to B(\mathcal{H}_i)$, $i = 1, 2$, there is an induced representation*

$$\pi_1 \odot \pi_2 : A_1 \odot A_2 \to B(\mathcal{H}_1 \otimes \mathcal{H}_2)$$

such that $\pi_1 \odot \pi_2(a_1 \odot a_2) = \pi_1(a_1) \otimes \pi_2(a_2)$ for all $a_i \in A_i$, $i = 1, 2$.

We have discussed extending pairs of linear maps to tensor products, but what about restricting maps on tensor products to the tensor factors? Given a $*$-homomorphism on an algebraic tensor product of C*-algebras $\phi : A \odot B \to C$, when can we define restrictions $\phi|_A : A \to C$ and $\phi|_B : B \to C$? In general this is not so easy. In the unital setting, there is a natural way to do this.

Exercise 12.5. Suppose A, B, and C are C*-algebras with A and B unital and $\phi : A \odot B \to C$ a $*$-homomorphism. Then there exist $*$-homomorphisms $\phi_A : A \to C$ and $\phi_B : B \to C$ with commuting ranges such that $\phi = \phi_A \times \phi_B$.

A little harder to prove is the following (without the assumption that A and B are unital). See [5, Theorem 3.6.2].

Theorem 12.6. *Let A and B be C*-algebras and $\pi : A \odot B \to B(\mathcal{H})$ a nondegenerate $*$-homomorphism. Then there exist nondegenerate representations $\pi_A : A \to B(\mathcal{H})$ and $\pi_B : B \to B(\mathcal{H})$ so that $\pi = \pi_A \times \pi_B$.*

Exercise 12.7. How would you define the representations when A_1 and A_2 are unital? Given a representation $\pi : A_1 \odot A_2 \to B(\mathcal{H})$, show that the restrictions $\pi_i : A_i \to B(\mathcal{H})$ have commuting images.

12.1. **C*-norms on tensor products.** For C*-algebras A and B, $A \odot B$ is a $*$-algebra. In order to turn it into a C*-algebra, we need to be able to define a C*-norm $\|\cdot\|$ on $A \odot B$. With this, $(A \odot B, \|\cdot\|)$ will be a *pre-*C**-algebra*, i.e. its completion is a C*-algebra. Much like the situation with groups, we are guaranteed the following:

- C*-norms on algebraic tensor products of C*-algebras always exist;
- there can be (very) many different C*-norms on a given algebraic tensor product of two C*-algebras;
- but we know how to describe the largest;
- and we have a nice canonical spatial norm (which unlike for groups is even the smallest!)[21]; and
- it is extremely interesting to ask when the two coincide (and this is related to the notion of amenability for groups because math is beautiful).

Definition 12.8. For C*-algebras A and B, a *cross norm* on a $A \odot B$ is a norm $\|\cdot\|$ such that for simple tensors we have $\|a \otimes b\| = \|a\|\|b\|$ for every $a \in A$ and $b \in B$.

Example 12.9. We verified in the prerequisite notes that for $T_1 \in B(\mathcal{H}_1)$ and $T_2 \in B(\mathcal{H}_2)$, the norm on $B(\mathcal{H}_1) \odot B(\mathcal{H}_2)$ inherited from $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is a cross norm. In fact as a consequence of Takesaki's theorem[22] (which we will discuss more later in this section) every C*-norm on $A \odot B$ is a cross norm. We will take this as a fact as we proceed.

[21] This is a deep theorem due to Takesaki.

[22] Full disclosure, using this theorem is wayyyy overkill. A functional calculus argument could prove this, but this section is already long enough.

In Exercise 12.2, we saw that there is an algebraic $*$-isomorphism $\mathrm{M}_n(\mathbb{C}) \odot A \cong \mathrm{M}_n(A)$, the latter being a C*-algebra with norm induced by the norm of A. Hence pulling back the norm along this $*$-isomorphism gives a C*-norm on $\mathrm{M}_n(\mathbb{C}) \odot A$ (i.e. $\|[\lambda_{ij}] \odot a\| = \|[\lambda_{ij} a]\|$). Moreover, $\mathrm{M}_n(\mathbb{C}) \odot A$ is already complete with respect to this norm, which means it is a C*-algebra. Hence any other C*-norm we define on $\mathrm{M}_n(A)$ agrees with this norm. (See remarks after Proposition 1.29.) That means we have proved the following proposition.

Proposition 12.10. *Let A be a C*-algebra and $1 \leq n < \infty$. Then there is a unique C*-norm on the algebraic tensor product $\mathrm{M}_n(\mathbb{C}) \odot A$, which comes from the $*$-isomorphism $\mathrm{M}_n(\mathbb{C}) \odot A \cong \mathrm{M}_n(A)$. Hence we write $\mathrm{M}_n(\mathbb{C}) \otimes A$.*

This identification also introduces very convenient notation, e.g. for the diagonal matrix in $\mathrm{M}_n(A)$ with $a \in A$ down the diagonal:

$$I_n \otimes a \;\leftrightarrow\; \begin{bmatrix} a & 0 & \dots & 0 \\ 0 & a & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & a \end{bmatrix}.$$

For general C*-algebras A and B, it should not be taken for granted that a C*-norm exists at all on $A \odot B$. However, it turns out the two most natural candidates both yield C*-norms.

The first is the spatial norm, i.e. the norm inherited as a subspace of bounded operators on a tensor product of Hilbert spaces. Recall that as a consequence of the GNS construction, every C*-algebra has at least one faithful representation on some Hilbert space.

Definition 12.11 (Spatial Norm)**.** Let $\pi_i : A_i \to B(\mathcal{H}_i)$ be faithful representations. The *spatial* norm on $A_1 \odot A_2$ is

$$\left\| \sum a_i \odot b_i \right\|_{\min} = \left\| \sum \pi_1(a_i) \otimes \pi_2(b_i) \right\|_{B(\mathcal{H}_1 \otimes \mathcal{H}_2)}.$$

Remark 12.12. We will explain the $\|\cdot\|_{\min}$ notation later with Takesaki's theorem, which we keep mentioning.

Exercise 12.13. Check that $\|\cdot\|_{\min}$ is a semi-norm satisfying the C*-identity.

Proposition 12.14. *The semi-norm $\|\cdot\|_{\min}$ is a norm, i.e. for each $x \in A_1 \odot A_2$, if $\|x\|_{\min} = 0$, then $x = 0$.*

Proof. Let $\pi_i : A_i \to B(\mathcal{H}_i)$ be faithful representations. Then the algebraic tensor product map $\pi_1 \odot \pi_2 : A_1 \odot A_2 \to B(\mathcal{H}_1) \odot B(\mathcal{H}_2)$ is injective. By Proposition 4.23, we can view $B(\mathcal{H}_1) \odot B(\mathcal{H}_2)$ as a $*$-subalgebra of $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, and consequently have $\pi_1 \odot \pi_2 : A_1 \odot A_2 \to B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ injective. Then for any $x = \sum_{i=1}^n a_i \odot b_i \in A_1 \odot A_2$, if $\|x\|_{\min} = 0$, then

$$0 = \|x\|_{\min} = \|\sum_{i=1}^{n} \pi_1(a_i) \otimes \pi_2(b_i)\| = \|(\pi_1 \odot \pi_2)(x)\|,$$

which by injectivity means $x = 0$. ☐

Hence $\|\cdot\|_{\min}$ is a norm, and we can define the C*-algebra

$$A \otimes B := \overline{A \odot B}^{\|\cdot\|_{\min}}.$$

It is sometimes denoted $A \otimes_{\min} B$, but we choose the undecorated notation to match the literature. In most cases this the unofficial "default" norm to take on a tensor product of C*-algebras.[23]

For a sense of perspective, dropping the representation notation, we view $A_1 \subset B(\mathcal{H}_1)$ and $A_2 \subset B(\mathcal{H}_2)$. Then there is a natural way to stick them into a common C*-algebra, i.e. $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, from whence they can inherit the C*-norm, i.e. $A_1 \otimes A_2$ is the closure of the $*$-subalgebra $A_1 \odot A_2 \subset B(\mathcal{H}_1 \otimes \mathcal{H}_2)$.

However, the norm was defined with an arbitrary choice of faithful representations. Fortunately, the value of the norm is independent of that choice.

Proposition 12.15. *Given faithful representations $\pi_i : A_i \to B(\mathcal{H}_i)$ and $\pi_i' : A_i \to B(\mathcal{H}_i')$, then the minimal tensor norms $\|\cdot\|_{\min}$ and $\|\cdot\|_{\min}'$ defined by each pair of faithful representations agree.*

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[23]For groups, it's the other way around and the maximal C*-completion of the group algebra is often the undecorated one.

The proof is nice to see because it highlights two useful techniques. The first, yet again, is approximate identities. The second is the fact that there is only one C*-norm on $M_n(B)$ for any C*-algebra B.

In our proof, we limit ourselves to the countable setting to avoid the extra notation involved with nets.

Proof. By symmetry, it suffices to prove the case where $\mathcal{H}_1 = \mathcal{H}_1'$ and $\pi_1 = \pi_1'$.

We first consider the case where $A_1 = M_n(\mathbb{C})$ for some n. Since both $\|\cdot\|_{\min}$ and $\|\cdot\|_{\min}'$ are C*-norms, by Proposition 12.10, for every $x = \sum_{i=1}^m T_i \odot a_i \in M_n(\mathbb{C}) \odot A_2$,

$$\left\| \sum_{i=1}^n \pi_1(T_i) \otimes \pi_2(a_i) \right\| = \|x\|_{\min} = \|x\|_{\min}' = \left\| \sum_{i=1}^n \pi_1(T_i) \otimes \pi_2'(a_i) \right\|. \tag{12.2}$$

Now, for the general separable case, take an increasing net of finite-rank projections $P_1 \leq P_2 \leq \ldots$ in $B(\mathcal{H}_1)$ where the rank of P_n is n and such that $\|P_n\xi - \xi\| \to 0$ for all $\xi \in \mathcal{H}_1$ (i.e. P_n converge in SOT to $1_{\mathcal{H}_1}$). Then for every $T \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, $(P_n \otimes 1_{\mathcal{H}_2})T(P_n \otimes 1_{\mathcal{H}_2})$ converges in $*$-SOT[24] to T, and so we have (check)

$$\|T\| = \sup_n \|(P_n \otimes 1_{\mathcal{H}_2})T(P_n \otimes 1_{\mathcal{H}_2})\|.$$

That means for any $x = \sum_{i=1}^m a_i \odot b_i \in A_1 \odot A_2$,

$$\|x\|_{\min} = \sup_n \left\| \sum_{i=1}^m P_n \pi(a_i) P_n \otimes \pi_2(b_i) \right\|$$
$$\|x\|_{\min}' = \sup_n \left\| \sum_{i=1}^m P_n \pi(a_i) P_n \otimes \pi_2'(b_i) \right\|.$$

For $n \in \mathbb{N}$, define a $*$-isomorphism $\phi : M_n(\mathbb{C}) \to P_n B(\mathcal{H}) P_n$. Since ϕ is a faithful representation of $M_n(\mathbb{C})$, by (12.2), we have

$$\begin{aligned}\left\| \sum_{i=1}^m P_n \pi(a_i) P_n \otimes \pi_2(b_i) \right\| &= \left\| \sum_{i=1}^m \phi(\phi^{-1}(P_n \pi(a_i) P_n)) \otimes \pi_2(b_i) \right\| \\ &= \left\| \sum_{i=1}^m \phi(\phi^{-1}(P_n \pi(a_i) P_n)) \otimes \pi_2'(b_i) \right\| \\ &= \left\| \sum_{i=1}^m P_n \pi(a_i) P_n \otimes \pi_2'(b_i) \right\|.\end{aligned}$$

It follows that $\|x\|_{\min} = \|x\|_{\min}'$. □

So, given C*-algebras A_1 and A_2 and faithful nondegenerate representations $\pi_i : A_i \to B(\mathcal{H}_i)$, we complete $\pi_1 \odot \pi_2$ to a faithful representation

$$\pi_1 \otimes \pi_2 : A_1 \otimes A_2 \to B(\mathcal{H}_1 \otimes \mathcal{H}_2).$$

There is another often useful description of the minimal tensor norm.

Proposition 12.16. *For C*-algebras A_1 and A_2, and $x = \sum_{j=1}^n a_j \odot b_j \in A_1 \odot A_2$,*

$$\|x\|_{\min} = \sup\{\| \sum_{j=1}^n \pi_1(a_j) \otimes \pi_2(b_j)\| \ : \ \pi_i : A_i \to B(\mathcal{H}_i) \text{ (nondegenerate) representations}\}.$$

Proof. Let $\pi_i : A_i \to B(\mathcal{H}_i)$ be representations and $\sigma_i : A_i \to B(\mathcal{H}_i')$ be faithful representations. Then by Exercise 4.29, $\pi_i \oplus \sigma_i : A_i \to B(\mathcal{H}_i \oplus \mathcal{H}_i')$ is a faithful representation. Let $P_i \in B(\mathcal{H}_i \oplus \mathcal{H}_i')$ be the compression to $\mathcal{H}_i$ for each $i = 1, 2 \ldots$ □

Exercise 12.17. Finish the proof of Proposition 12.16. This is an example of a technique where one can *dilate* a map to one with a desired property (e.g. faithfulness) and then *cut down* to the original map to draw the desired conclusion.

[24] $S_n \to S$ in $*$-SOT if $S_n \to S$ in SOT and $S_n^* \to S^*$ in SOT.

Corollary 12.18. *For a pair of $*$-homomorphisms $\phi_i : A_i \to B_i$, the algebraic tensor product $\phi_1 \odot \phi_2$ extends to a $*$-homomorphism*

$$\phi_1 \otimes_{\min} \phi_2 : A_1 \otimes_{\min} A_2 \to B_1 \otimes_{\min} B_2.$$

Proof. We are charged with showing that $\phi_1 \odot \phi_2$ is continuous with respect to the topologies on $A_1 \odot A_2$ and $B_1 \odot B_2$ induced by their respective $\|\cdot\|_{\min}$ norms. We know that there exist faithful representations $\pi_i^A : A_i \to B(\mathcal{H}_i^A)$ and faithful representations $\pi_i^B : B_i \to B(\mathcal{H}_i^B)$. So if $x = \sum_{j=1}^n a_j \odot b_j \in A_1 \odot A_2$, the fact that $*$-homomorphisms are norm-decreasing means that

$$\|x\|_{A_1 \otimes_{\min} A_2} = \|\sum_{j=1}^n \pi_1^A(a_j) \otimes \pi_2^A(b_j)\| \geq \|\sum_{j=1}^n \pi_1^B(\phi_1(a_j)) \otimes \pi_2^B(\phi_2(b_j))\| = \|\phi_1 \odot \phi_2(x)\|_{B_1 \otimes_{\min} B_2}.$$

But each $\pi_i^B \circ \phi_i : A_i \to B(\mathcal{H}_i^B)$ is a representation of A_i, so we complete the proof via an appeal to the preceding proposition. □

Just as with groups, there is another natural norm which comes from taking all possible representations.

Definition 12.19 (Maximal Norm)**.** Let A and B be C*-algebras. We define the maximal C*-tensor norm on $A \odot B$ by

$$\|x\|_{\max} = \sup\{\|\pi(x)\| : \pi : A \odot B \to B(\mathcal{H}) \text{ a (nondegenerate) rep}\}$$

for all $x \in A \odot B$.

The first question is if this is even finite; it is by Theorem 12.6. Indeed, given $\pi : A \odot B \to B(\mathcal{H})$, with restrictions $\pi|_A$ and $\pi|_B$, then we have for all simple tensors $a \odot b \in A \odot B$,

$$\|\pi(a \odot b)\| = \|\pi|_A(a)\pi|_B(b)\| \leq \|\pi|_A(a)\|\|\pi|_B(b)\| \leq \|a\|\|b\| < \infty.$$

Just as we argued for groups (Proposition 5.10), this with the triangle inequality guarantees that $\|x\|_{\max} < \infty$ for all $x \in A \odot B$.

Exercise 12.20. Check that $\|\cdot\|_{\max}$ is a semi-norm satisfying the C*-identity.

For any pair of faithful representations $\pi_i : A_i \to B(\mathcal{H}_i)$, we get a representation $\pi = \pi_1 \odot \pi_2 : A_1 \odot A_2 \to B(\mathcal{H}_1 \otimes \mathcal{H}_2)$. It follows that for any $x \in A_1 \odot A_2$,

$$\|x\|_{\min} = \|\pi(x)\| \leq \|x\|_{\max}.$$

So, for any $x \in A_1 \odot A_2$,

$$\|x\|_{\max} = 0 \Rightarrow \|x\|_{\min} = 0 \Rightarrow x = 0,$$

which means $\|\cdot\|_{\max}$ is a norm. Hence we define the C*-algebra

$$A_1 \otimes_{\max} A_2 := \overline{A_1 \odot A_2}^{\|\cdot\|_{\max}}.$$

Remark 12.21. Note that by definition, the $*$-algebra $A_1 \odot A_2$ is a dense subalgebra in $A_1 \otimes_{\max} A_2$ and $A_1 \otimes A_2$.

Just as with groups, the maximal tensor product enjoys the following universal property.

Proposition 12.22. *If $\phi : A_1 \odot A_2 \to C$ is a $*$-homomorphism, then there exists a unique $*$-homomorphism $A_1 \otimes_{\max} A_2 \to C$ which extends ϕ. In particular, any pair of $*$-homomorphisms $\phi_i : A_i \to C$ with commuting ranges induces a unique $*$-homomorphism*

$$\phi_1 \times \phi_2 : A_1 \otimes_{\max} A_2 \to C.$$

Note that this is really just a statement about norms, and it is a theme we've seen before (Proposition 5.10). Let's flesh out a more general idea that underlies both.

Suppose B and C are C*-algebras, $B_0 \subset B$ is a dense $*$-subalgebra, and $\pi : B_0 \to C$ is a $*$-homomorphism. (Notice that, unless $B_0 = B$, this means B_0 is *not* a C*-algebra.) The only obstruction to extending π to a $*$-homorphism on B is if π is not contractive on B_0, i.e. $\|\pi(b)\| > \|b\|$ for some $b \in B_0$. In other words, π extends to B iff π is contractive on B_0. The necessity is easy to see. Indeed, if π does extend to B, then the C*-norm on B forces π to be contractive on all of B, including B_0. On the other hand, if $\pi : B_0 \to C$ is a contractive $*$-homomorphism, then it is in particular bounded, which means it extends to a bounded homomorphism $\pi : B \to C$. Moreover, just as we saw in Proposition 5.10, for any $b \in B$ with

$b_n \in B_0$ converging to b, we have $\pi(b_n) \to \pi(b)$ and hence $\pi(b_n)^* \to \pi(b)^*$. Then by uniqueness of limits, $\pi(b^*) = \pi(b)^*$ since

$$\|\pi(b_n)^* - \pi(b^*)\| = \|\pi(b_n^*) - \pi(b^*)\| \to 0.$$

For the sake of reference, we record this in a proposition:

Proposition 12.23. *Suppose B and C are* C*-*algebras, $B_0 \subset B$ is a dense $*$-subalgebra, and $\pi : B_0 \to C$ is a $*$-homomorphism. Then π extends to B iff π is contractive on B_0.*

With that digression, the proof of proposition 12.22 is quite immediate.

Proof of Proposition 12.22. Take a faithful nondegenerate representation $\pi : C \to B(\mathcal{H})$. Then $\pi \circ \phi : A_1 \odot A_2 \to B(\mathcal{H})$ is a contractive $*$-homomorphism (with respect to the $\|\cdot\|_{\max}$ norm) and hence extends to $A \otimes_{\max} A_2$. □

It follows from this that $\|\cdot\|_{\max}$ is the largest possible C*-norm on $A_1 \odot A_2$.

Corollary 12.24. *Given any* C*-*norm $\|\cdot\|$ on $A_1 \odot A_2$, there is a surjective $*$-homomorphism $A_1 \otimes_{\max} A_2 \to \overline{A_1 \odot A_2}^{\|\cdot\|}$ extending the identity map on $A_1 \odot A_2$.*

Proof. Suppose $\|\cdot\|$ is another C*-norm on $A_1 \odot A_2$. Then the identity map $A_1 \odot A_2 \to \overline{A_1 \odot A_2}^{\|\cdot\|}$ is a $*$-homomorphism, which then extends to a $*$-homorphism

$$A_1 \otimes_{\max} A_2 \to \overline{A_1 \odot A_2}^{\|\cdot\|}.$$

Since it is a $*$-homomorphism, its image is closed and contains the dense subset $A_1 \odot A_2$, and so it is a surjection. As a surjective $*$-homomorphism, it is contractive, and so $\|x\|_{\max} \geq \|x\|$ for all $x \in A_1 \odot A_2$. □

Remark 12.25. Very often in the literature, the closure of $A \odot B$ with respect to an arbitrary tensor norm is denoted by $A \otimes_\alpha B$ where the norm is denoted by $\|\cdot\|_\alpha$.

It turns out that the spatial norm $\|\cdot\|_{\min}$ *is* the minimal C*-norm on $A_1 \odot A_2$. This is an important theorem due to Takesaki whose proof involves some heavy work in extending states to tensor products. For the sake of time, we will have to take this for granted. The proof is worked out in [5, Section 3].

Theorem 12.26 (Takesaki)**.** *The spatial norm $\|\cdot\|_{\min}$ is the minimal* C*-*norm on $A_1 \odot A_2$. In other words, given any* C*-*norm $\|\cdot\|$ on $A_1 \odot A_2$, there are surjective $*$-homomorphisms*

$$A_1 \otimes_{\max} A_2 \to \overline{A_1 \odot A_2}^{\|\cdot\|} \to A_1 \otimes A_2$$

extending the identity map

$$A_1 \odot A_2 \to A_1 \odot A_2 \to A_1 \odot A_2.$$

It follows that if the *natural* surjection $A_1 \otimes_{\max} A_2 \to A_1 \otimes A_2$ is injective, then $A_1 \odot A_2$ has a unique tensor norm. This fact is often indicated by writing

$$A_1 \otimes_{\max} A_2 = A_1 \otimes A_2.$$

Remark 12.27. It is important here that it is this natural surjection that is also injective, i.e. the one that extends the identity map $A_1 \odot A_2$.

We have been avoiding the non-unital elephant in the room. We relegate the proof to [5, Corollary 3.3.12].

Proposition 12.28. *If A and B are* C*-*algebras with A non-unital, then any* C*-*norm on $A \odot B$ can be extended to a* C*-*norm on $\tilde{A} \odot B$ (meaning the norms agree on $A \odot B \subset \tilde{A} \odot B$). Similarly, when both A and B are non-unital, any* C*-*norm can be extended to $\tilde{A} \odot \tilde{B}$.*[25]

Exercise 12.29. For C*-algebras A and B, we have canonical[26] isomorphisms $A \otimes B \cong B \otimes A$ and $A \otimes_{\max} B \cong B \otimes_{\max} A$.

Exercise 12.30. Consider the crossed product of a C^*-algebra A by the trivial action of a discrete group G – that is, $\alpha_g(a) = a$ for all $g \in G, a \in A$. Show that $A \rtimes_{\alpha,r} G \cong A \otimes C_r^*(G)$ and $A \rtimes_\alpha G \cong A \otimes_{\max} C^*(G)$.

[25] In general (i.e. when we don't have $A = \tilde{A}$ or $B = \tilde{B}$, this is a larger algebra than $\widetilde{A \odot B}$.

[26] i.e. This is another way of saying "natural". In this setting, this means the maps extend the usual algebraic maps.

12.2. **Inclusions and Short Exact Sequences.** This section is dedicated to two properties that held automatically for algebraic tensor products but that can now fail for their C*-completions:

(1) They respect inclusions, i.e. if B and C are C*-algebras and $A \subset B$ a C*-subalgebra, then we have a natural inclusion
$$A \odot C \hookrightarrow B \odot C.$$
(2) They respect exact sequences, i.e. if B and C are C*-algebras and $J \lhd B$ an ideal, then the following sequence is exact.
$$0 \to J \odot C \to B \odot C \to B/J \odot C \to 0.$$

Proposition 12.31. *Let B and C be* C*-*algebras, $A \subset B$ a* C*-*subalgebra. Then*

(1) We have a natural inclusion $A \otimes_{\min} C \subseteq B \otimes_{\min} C$.
(2) This can fail for the maximal tensor product.

Exercise 12.32. Check (1). (This is just a statement about norms on sums of simple tensors.)

For (2), that's where things get interesting. Questions about embeddability of maximal tensor products get hard quick. So, it's easiest to explain why it can go wrong. Recall that the maximal tensor product norm was defined as a supremum over all representations. A representation on $B \odot C$ restricts to one on $A \odot C$, but a representation on $A \odot C$ need not extend to $B \odot C$. So, in general the sup taken for the maximal norm on $A \odot C$ is taken over a larger set than the one for $B \odot C$.

Remark 12.33. One fact that will play a role promptly is that this *does* hold when A is an ideal in B. A representation from an ideal $J \lhd A$ in a C*-algebra does always extend to a representation on A (see [1, Section 1.3]). So when $J \lhd A$ is an ideal, then so is $J \odot C$ for any C*-algebra C, and we have $J \otimes_{\max} C \lhd A \otimes_{\max} C$.

Here are some examples of where this can go wrong. Unfortunately, we haven't built up sufficient terminology to explain the details.

Example 12.34. In his work leading up to a remarkable characterization of Connes Embedding Problem, Kirchberg characterized Lance's Weak Expectation Property (WEP) (see [5, Exercise 2.3.14]) as follows: A C*-algebra B has WEP if and only if there is a unique C*-tensor norm on $B \odot \mathrm{C}^*(\mathbb{F}_2)$. It follows that if we have a C*-algebra B with WEP and a C*-subalgebra $A \subset B$ without WEP, then $A \otimes_{\max} \mathrm{C}^*(\mathbb{F}_2)$ does not embed into $B \otimes_{\max} \mathrm{C}^*(\mathbb{F}_2)$.

Examples of C*-algebras with WEP include all nuclear C*-algeras as well as $B(\mathcal{H})$ for any Hilbert space $\mathcal{H}$. What are examples of such pairs?

Using Kirchberg's $\mathcal{O}_2$ embedding theorem (a very difficult and sophisticated result in C*-theory), we know that all separable exact C*-algebras embed into the nuclear C*-algebra $\mathcal{O}_2$.[27] Back when he introduced WEP, Lance proved (and you can too; see [5, Exercise 2.3.14]) that if a separable exact C*-algebra has WEP, then it is nuclear. So, if A is separable exact and non-nuclear, then $A \otimes_{\max} \mathrm{C}^*(\mathbb{F}_2)$ does not embed into $\mathcal{O}_2 \otimes_{\max} \mathrm{C}^*(\mathbb{F}_2)$. A nice (sophisticated) example due to Wasserman of an exact non-nuclear C*-algebra is $\mathrm{C}^*_r(\mathbb{F}_2)$.[28]

If you believe the announced refutations of Connes Embedding Problem, then $\mathrm{C}^*(\mathbb{F}_2)$ does not have WEP. As a separable C*-algebra, it embeds into $B(\mathcal{H})$, and so $\mathrm{C}^*(\mathbb{F}_2) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_2)$ does not embed into $B(\mathcal{H}) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_2)$.

In the above, $\mathrm{C}^*(\mathbb{F}_2)$ can be replaced by other "sufficiently large" C*-algebras.

Proposition 12.35. *Let B and C be* C*-*algebras and $J \lhd B$ an ideal. Then*

(1) The sequence
$$0 \to J \otimes_{\max} C \to B \otimes_{\max} C \to B/J \otimes_{\max} C \to 0$$
is exact.
(2) This can fail for the minimal (i.e. spatial) tensor product.

[27] Remember, kids, although amenability passes to subgroups, nuclearity does **not** pass to subalgebras! In fact, this is a favorite example demonstrating the fact!

[28] Fyi: $\mathrm{C}^*(\mathbb{F}_2)$ is non-exact, basically because singly generated unital non-exact C*-algebras exist, exactness passes to quotients (one of the deepest theorems in C*-algebras, also due to Kirchberg), and $\mathrm{C}^*(\mathbb{F}_2)$ surjects onto any singly generated unital C*-algebra (Exercise: use functional calculus, in particular that e^{ia} is a unitary for any $a = a^*$, and the universality of $\mathrm{C}^*(\mathbb{F}_2)$ to check the last bit).

For (1), the proof in full detail is provided in [5, Proposition 3.7.1]. We simply give an idea of what needs to be shown. In either case, $J \otimes_{\max} C \lhd B \otimes_{\max} C$ and $J \otimes C \lhd B \otimes C$. So we have exact sequences

$$0 \to J \otimes_{\max} C \to B \otimes_{\max} C \to (B \otimes_{\max} C)/(J \otimes_{\max} C) \to 0$$

and

$$0 \to J \otimes_{\min} C \to B \otimes_{\min} C \to (B \otimes_{\min} C)/(J \otimes_{\min} C) \to 0.$$

In both cases, from the algebraic identification $B/J \odot C \cong (B \odot C)/(B \odot J)$ one argues that there is a C*-norm so that

$$(B \otimes_{\max} C)/(J \otimes_{\max} C) \cong B/J \otimes_{\alpha} C \quad \textit{and} \quad (B \otimes_{\min} C)/(J \otimes_{\min} C) = B/J \otimes_{\beta} C.$$

It will follow from the maximality of $\|\cdot\|_{\max}$ that $\otimes_\alpha = \otimes_{\max}$. But for the other quotient, that won't always happen.

Definition 12.36. A C*-algebra C is *exact* if the sequence

$$0 \to J \otimes_{\min} C \to B \otimes_{\min} C \to (B \otimes_{\min} C)/(J \otimes_{\min} C) \to 0$$

is exact for any C*-algebra B and any ideal $J \lhd B$.

Though seemingly unrelated, the two definitions we have given for exactness are indeed equivalent, though the proof of this is not easy.

Theorem 12.37 (Kirchberg)**.** *A* C*-*algebra is exact in the sense of Definition 11.17 if and only if the functor* $\otimes_{\min} A$ *is exact, i.e. if the above definition holds.*

The question of when two C*-algebras have a unique C*-tensor norm is very difficult, and resolving this question for certain algebras is equivalent to resolving big open problems.

For example, thanks to deep and groundbreaking work of Kirchberg, we know that a famous recently-resolved problem, Connes' Embedding Problem, is equivalent to answering the question of whether or not $C^*(\mathbb{F}_2) \otimes_{\max} C^*(\mathbb{F}_2) = C^*(\mathbb{F}_2) \otimes C^*(\mathbb{F}_2)$. (Ask Brent and Rolando for the the original statement.) Further work (building on Kirchberg's results) connected this to what is known as Tsirelson's problem in quantum information theory, which was what was actually refuted last year.

Another example is A. Thom's example of a hyperlinear group that is not residually finite. (Again, thanks to work of Kirchberg, this is equivalent to the full group C*-algebra of said group not having a unique tensor norm with $B(\mathcal{H})$.)

Another example is Junge and Pisier's proof that $B(\mathcal{H}) \odot B(\mathcal{H})$ does not have a unique C*-tensor norm when $\mathcal{H}$ is infinite dimensional, which was proven by Kirchberg to be equivalent to another collection of open problems.

Remark 12.38. You may have noticed that Kirchberg was very influential in a lot of results pertaining to tensor products of C*-algebras. Yeah.

Remark 12.39 (Remark on tensors and commutativity). Given C*-algebras A_1 and A_1, an example of a representation of $A_1 \odot A_2 \to B(\mathcal{H})$ is the tensor product of two representations,

$$\sigma_1 \odot \sigma_2 : A_1 \odot A_2 \to B(\mathcal{H}_1 \otimes \mathcal{H}_2).$$

But in general, there can be many representations that are not of this form, i.e. for some $x \in A_1 \odot A_2$, we could have

$$\begin{aligned} \|x\|_{\max} &= \sup\{\|\pi(x)\| \ : \pi : A_1 \odot A_2 \to B(\mathcal{H})\} \\ &> \sup\{\|\pi_1 \odot \pi_2(x)\| \ : \pi_i : A_i \to B(\mathcal{H}_i)\}. \end{aligned}$$

On a philosophical level, this is a question about commutativity. Given C*-algebras A_1 and A_2, is there any context (= C*-algebra they can be simultaneously embedded into) where A_1 and A_2 commute but *not* as tensors. Let's try to flesh this out a little.

Given a representation $\pi : A_1 \odot A_2 \to B(\mathcal{H})$, the restrictions $\pi_i : A_i \to B(\mathcal{H})$ have commuting images (Exercise 12.7). When $\pi = \sigma_1 \odot \sigma_2 : A_1 \odot A_2 \to B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, we have a much better idea of what the images are and why they commute. In this case the restrictions are given for $a_i \in A_i$ by

$$\pi_1(a_1) = \sigma_1(a_1) \otimes 1_{\mathcal{H}_1} \quad \text{and} \quad \pi_2(a_2) = 1_{\mathcal{H}_2} \otimes \sigma_2(a_2).$$

Then we have

$$\pi_1(a_1)\pi_2(a_2) = \big(\sigma_1(a_1)\otimes 1_{\mathcal{H}_1}\big)\big(1_{\mathcal{H}_2}\otimes\sigma_2(a_2)\big) = \sigma_1(a_1)\otimes\sigma_2(a_2) = \big(1_{\mathcal{H}_2}\otimes\sigma_2(a_2)\big)\big(\sigma_1(a_1)\otimes 1_{\mathcal{H}_1}\big) = \pi_2(a_2)\pi_1(a_1).$$

12.3. **Nuclearity.** On the other end of the spectrum are C*-algebras which always have unique tensor product norms. The term originally used for such C*-algebras was in fact "nuclear." But we've already used this term for C*-algebras satisfying the completely positive approximation property. That these two coincide is a remarkable theorem, independently proved by Choi-Effros and Kirchberg

Theorem 12.40 (Choi-Effros, Kirchberg)**.** *A* C**-algebra A satisfies the completely positive approximation property (Definition 11.6) if and only if $A \odot B$ has a unique* C**-tensor norm for any* C**-algebra B.*

The proof of this theorem would require us to build up a fair bit of theory first, so we simply point you to Chapters 2 and 3 in [5], where the argument and surrounding theory is laid out quite well.

In general, it's often easier to prove that a C*-algebra has the completely positive approximation property (an internal property) as opposed to always having a unique tensor product norm (an external property). However, it was not so hard to show the latter for one class of C*-algebras.

Example 12.41. From Proposition 12.10, we know that $\mathrm{M}_n(\mathbb{C})$ is nuclear for any $n \in \mathbb{N}$. It turns out that any finite-dimensional C*-algebra is nuclear. (This mostly comes down to Proposition 8.5. See [11, Theorem 6.3.9] for more details.)

We have already seen that $K(\mathcal{H})$, as an AF algebra, is nuclear. Just for fun, here's an argument from the tensor product perspective.

Example 12.42. Let $\mathcal{K}$ denote the compact operators on some Hilbert space $\mathcal{H}$ and A any C*-algebra.

First we claim that $FR(\mathcal{H}) \odot A$ is a dense $*$-subalgebra of $\mathcal{K} \odot A$ with respect to any C*-norm on $\mathcal{K} \odot A$. We know from Day 1 lectures that $FR(\mathcal{H})$ is dense in $\mathcal{K}$. Now, suppose $S \odot a \in \mathcal{K} \odot A$ and $S_j \in FR(\mathcal{H})$ a sequence with $S_j \to S$. Recall that any C*-norm $\|\cdot\|$ on $\mathcal{K} \odot A$ is a cross norm, and so for any C*-norm $\|\cdot\|$ on $\mathcal{K} \odot A$, we have

$$\|(S \odot a) - (S_j \odot a)\| = \|(S - S_j) \odot a\| = \|S - S_j\|\|a\| \to 0.$$

Using the triangle inequality, we can extend this to show that any $x = \sum_{j=1}^m T_j \odot a_j \in \mathcal{K} \odot A$ can be approximated in any C*-norm by sums of simple tensors of finite rank operators.

So if we know $\|x\|_{\max} = \|x\|_{\min}$ for any $x \in FR(\mathcal{H}) \odot A$, then it follows that the natural surjection $\mathcal{K} \otimes_{\max} A \to \mathcal{K} \otimes A$ is isometric and $\mathcal{K}$ is nuclear. Fix an arbitrary $x = \sum_{j=1}^m T_j \odot a_j \in FR(\mathcal{H}) \odot A$, and let $\pi : \mathcal{K} \odot A \to B(\mathcal{H})$ be a representation. Then there exists a projection $P \in B(\mathcal{H})$ of rank $n < \infty$ such that $T_j = PT_jP$ for all j, and $x = \sum_{j=1}^m PT_jP \odot a_j$. Hence $x \in PB(\mathcal{H})P \odot A$. From Exercise 7.41 from Day 1 Lectures, we have a $*$-isomorphism $\phi : \mathrm{M}_n(\mathbb{C}) \to PB(\mathcal{H})P$, and hence a representation $\pi' := \pi \circ (\phi \odot id_A) : \mathrm{M}_n(\mathbb{C}) \odot A \to B(\mathcal{H})$.

Since we know $\mathrm{M}_n(\mathbb{C}) \otimes_{\max} A = \mathrm{M}_n(\mathbb{C}) \otimes_{\min} A$, we know that for any faithful representations $\sigma_1 : \mathrm{M}_n(\mathbb{C}) \to B(\mathcal{H}_1)$ and $\sigma_2 : A \to B(\mathcal{H}_2)$,

$$\begin{aligned}\|\sum_{j=1}^m \sigma_1(\phi^{-1}(PT_jP)) \odot \sigma_2(a_j)\|_{B(\mathcal{H}_1\otimes\mathcal{H}_2)} &= \|\sum_{j=1}^m \phi^{-1}(PT_jP) \odot a_j\|_{\min} \\ &= \|\sum_{j=1}^m \phi^{-1}(PT_jP) \odot a_j\|_{\max} \geq \|\pi'(\sum_{j=1}^m \phi^{-1}(PT_jP) \odot a_j)\| \\ &= \|\pi(\sum_{j=1}^m PT_jP \odot a_j)\| = \|\pi(x)\|.\end{aligned}$$

In particular, this holds for the faithful representations $\sigma_1 = \mathrm{id}_{\mathcal{K}} \circ \phi : \mathrm{M}_n(\mathbb{C}) \to PB(\mathcal{H})P \subset \mathcal{K} \hookrightarrow B(\mathcal{H})$ and any faithful representation σ_2 of A. But then we have

$$\begin{aligned}\|x\|_{\min} &= \|\sum_{j=1}^{m} \mathrm{id}_{\mathcal{K}}(T_j) \odot \sigma_2(a_j)\|_{B(\mathcal{H}\otimes\mathcal{H}_2)} \\ &= \|\sum_{j=1}^{m} \sigma_1(\phi^{-1}(PT_jP)) \odot \sigma_2(a_j)\|_{B(\mathcal{H}\otimes\mathcal{H}_2)} \\ &\geq \|\pi(x)\|.\end{aligned}$$

Since $\pi : \mathcal{K} \odot A \to B(\mathcal{H})$ was arbitrary, it follows that

$$\|x\|_{\min} \geq \|x\|_{\max},$$

which finishes the proof.

Remark 12.43. Consider $\mathcal{K} = \mathcal{K}(\ell^2)$. It follows from Example 12.42 that the completion of $\mathcal{K} \odot \mathcal{K}$ under any tensor norm can be identified with the completion of $\mathcal{K} \odot \mathcal{K}$ with respect to the norm on $B(\ell^2 \odot \ell^2)$ (via the tensor product of faithful representations $id_{\mathcal{K}} \odot id_{\mathcal{K}}$). This will be a closed two-sided ideal in $B(\ell^2 \odot \ell^2)$, which means it must be the compact operators $\mathcal{K}(\ell^2 \odot \ell^2)$. Moreover, after a permutation of the basis elements, we have $\ell^2 \otimes \ell^2 \cong \ell^2$. With this, one can then argue that $\mathcal{K} \otimes \mathcal{K} \cong \mathcal{K}$. More generally, a C*-algebra is *stable* if $A \otimes \mathcal{K} \cong A$. (Because of nuclearity, it does not matter what tensor product we choose.)

Since $\mathcal{K}$ is stable and since $(A \otimes \mathcal{K}) \otimes \mathcal{K} \cong A \otimes (\mathcal{K} \otimes \mathcal{K}) \cong A \otimes \mathcal{K}$ for any C*-algebra A[29], we call $A \otimes \mathcal{K}$ the *stabilization* of A. This is a basic object in many results and theories in C*-algebras, such as multiplier algebras, K-theory and classification, and is closely tied to Morita equivalence for C*-algebras, as we will see in section **??**. It turns out that the stabilization of A is very similar to A from the perspective of many C*-algebraic invariants, and so replacing A by its stabilization gives one more "wiggle room" for computations without affecting the underlying structure very much.

There is another fundamental class of nuclear C*-algebras: commutative C*-algebras. This was not so hard to prove with the completely positive approximation property definition of nuclearity (Proposition 11.10). Before the Choi-Effros/Kirchberg theorem, Takesaki showed that tensor products with commutative C*-algebras always have a unique C*-norm, but the proof was much more involved.

Theorem 12.44 (Takesaki)**.** *Let A and C be* C*-*algebras with C commutative. Then there is a unique* C*-*tensor norm on $C \odot A$.*

12.4. $C_0(X, A)$ **as tensor products.** Let us spend a little more time on this last class of nuclear C*-algebras. Recall from the Gelfand Naimark Theorem that any commutative C*-algebra is $*$-isomorphic to $C_0(X)$ for some locally compact Hausdorff space X. With this in mind look into another description of the tensor product of a C*-algebra with a commutative C*-algebra.

Definition 12.45. Let A be a C*-algebra and X a locally compact Hausdorff space (when X is not compact, we denote by $X \cup \{\infty\}$ its one point compactification). Just as we did for $A = \mathbb{C}$, we define

$$C_0(X, A) := \{f : X \cup \{\infty\} \to A \ : \ f \text{ continuous and } f(\infty) = 0\}.$$

When X is moreover compact, this is the same as $C(X, A)$.

Lemma 12.46. *Let A be a* C*-*algebra and X a locally compact Hausdorff space. Define the $*$-homomorphism $\phi : C_0(X) \odot A \to C_0(X, A)$ on simple tensors by $f \odot a \mapsto f(\cdot)a$. This gives a $*$-homomorphism, which then extends to a surjective $*$-homomorphism $C_0(X) \otimes_{\max} A \to C_0(X, A)$. Moreover, ϕ is injective on $C_0(X) \odot A$.*

The proof that the image of ϕ is dense in $C_0(X, A)$ is another example of a "partition of unity argument." We will give the argument from [11, Lemma 6.4.16] in the case where X is compact. The non-compact case amounts to identifying $C_0(X, A) = \{f \in C(X \cup \{\infty\}, A) \ : f(\infty) = 0\}$ (see [11, Lemma 6.4.16] for full details).

[29]In fact, the associativity for the minimal and maximal tensor product norms holds for all C*-algebras, i.e. for C*-algebras A, B, C, we have $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$ and $(A \otimes_{\max} B) \otimes_{\max} C \cong A \otimes_{\max} (B \otimes_{\max} C)$. This is normally an exercise, but we have plenty already.

Recall that we take for granted the fact from topology that, given any compact Hausdorff space X with open cover $U_1, ..., U_n$, there exist continuous functions $h_1, ..., h_n : X \to [0,1]$ so that $\text{supp}(h_j) \subset U_j$ and $\sum_j h_j = 1$. (See [Theorem 2.13, Rudin, Real and Complex Analysis].) This is a *partition of unity* subordinate to $U_1, ..., U_n$ (in fact a rather nice one).

Proof of Lemma 12.46. Since there is nothing surprising in checking that ϕ is a $*$-homomorphism, which by universality extends to a $*$-homomorphism on $C_0(X) \otimes_{\max} A$, we move straight to the questions of injectivity and surjectivity.

For the surjectivity argument, we assume X is compact (or work in its one point compactification as aforementioned). Since the image of a $*$-homomorphism from a C*-algebra is closed, it suffices to show that $C(X, A)$ is the closed linear span of functions of the form $f(\cdot)a$ for $f \in C(X)$ and $a \in A$. Let $g \in C(X, A)$ and $\varepsilon > 0$. Since X is compact and g continuous, $g(X)$ is compact, which means we can find a finite collection $a_1, ..., a_n \in g(X) \subset A$ so that $\{B_\varepsilon(a_j)\}_j$ covers $g(X)$, and hence $U_j = g^{-1}(B_\varepsilon(a_j))$ forms a finite open cover of X. Since X is compact, the aforementioned fact from topology tells us there exist continuous functions $h_j : X \to [0,1]$, $1 \leq j \leq n$ so that for each j, $\text{supp}(h_j) \subset U_j$ and $\sum_j h_j(x) = 1$ for all $x \in X$. Notice that, by our choice of U_j, that means that for each $x \in X$, either $h_j(x) = 0$ or $\|g(x) - a_j\| < \varepsilon$. Then we compute for each $x \in X$,

$$\begin{aligned}\|g(x) - \sum_j h_j(x)a_j\| &= \| \left(\sum_j h_j(x)\right) g(x) - \sum_j h_j(x)a_j\| \\ &= \|\sum_j h_j(x)(g(x) - a_j)\| \leq \sum_j h_j(x)\|g(x) - a_j\| \\ &\leq \sum_j h_j(x)\varepsilon = \varepsilon.\end{aligned}$$

This establishes our claim.

For injectivity, on $C_0(X) \odot A$, suppose $c = \sum_{j=1}^n f_j \odot a_j \in \ker(\phi)$ where $f_1, ..., f_n \in C_0(X)$ and $a_1, ..., a_n$ are linearly independent elements of A. Then $\phi(c) = 0$ implies that $\sum f_j(x)a_j = 0$ for all $x \in X$. But now these $f_j(x)$ are just complex numbers, and so the linear independence of the $a_1, ..., a_n$ implies that $f_j(x) = 0$ for each $1 \leq j \leq n$ and every $x \in X$. That means $f_1 = ... = f_n = 0$ and so $c = 0$. Hence ϕ is injective on $C_0(X) \odot A$. ☐

Theorem 12.47. *If A is a* C*-*algebra and X is a locally compact Hausdorff space, then for any* C*-*tensor norm, we have* $\overline{C_0(X) \odot A}^{\|\cdot\|} \cong C_0(X, A)$.

Proof. Since the map ϕ from Lemma 12.46 is injective, the pull-back of the norm from $C_0(X, A)$ (i.e. $\|c\| = \|\phi(c)\|$) gives a C*-norm on $C_0(X) \odot A$ (as opposed to just a semi-norm). By Theorem 12.44, there is a unique C*-tensor norm on $C_0(X) \odot A$, which means this norm agrees with $\|\cdot\|_{\max}$. Hence the surjective $*$-homomorphism $C_0(X) \otimes_{\max} A \to C_0(X, A)$ is isometric, and hence a $*$-isomorphism. By identifying $C_0(X) \otimes_{\max} A$ with the closure of $C_0(X) \odot A$ under any other C*-norm, the claim follows. ☐

Example 12.48. Three particularly interesting cases are when $X = [0,1]$, $X = (0,1]$, and $X = (0,1)$.[30] For a C*-algebra A, the *cone* over A is the C*-algebra

$$CA := C_0((0,1], A) = \{f : (0,1] \to A \ : \ f \text{ is continuous and } \lim_{t \to 0} f(t) = 0\},$$

and the *suspension*[31] over A is the C*-algebra,

$$SA := C_0((0,1), A) := \{f : (0,1) \to A \ : \ f \text{ is continuous and } \lim_{t \to 0} f(t) = 0 = \lim_{t \to 1} f(t)\}.$$

The suspension will become very important when we get to K-theory. It is also sometimes denoted by ΣA.

[30] Depending on how we like to define our functions these intervals are sometimes replaced with homeomorphic copies, e.g., sometimes $\mathbb{R}$ is used in place of $(0,1)$. This certainly makes the "∞" notation more natural!

[31] The terms "cone" and "suspension" are inspired by, but not quite the same as, the notions from topology, in case you are wondering.

12.5. **Continuous linear maps on tensor products.** In Takesaki's proof that $\|\cdot\|_{\min}$ is the smallest C*-norm, a delicate and crucial part of the argument is showing that states extend to tensor products, i.e. for $\phi_i \in S(A_i)$, $\phi_1 \odot \phi_2$ extends to a state on $\overline{A_1 \odot A_2}^{\|\cdot\|}$ for any C*-norm $\|\cdot\|$ (mapping into $\mathbb{C} \otimes \mathbb{C} = \mathbb{C}$).

Given a pair of $*$-homomorphisms $\phi_i : A_i \to B_i$, we have a $*$-homomorphism

$$\phi_1 \odot \phi_2 : A_1 \odot A_2 \to B_1 \odot B_2$$

defined on the dense $*$-subalgebra $A_1 \odot A_2$ of $\overline{A_1 \odot A_2}^{\|\cdot\|}$ where $\|\cdot\|$ is any C*-norm. By Proposition 12.23, this extends to a $*$-homomorphism on $\overline{A_1 \odot A_2}^{\|\cdot\|}$ iff $\phi_1 \odot \phi_2$ is contractive on sums of simple tensors. Naturally, this depends on the norm we put on $B_1 \odot B_2$ (e.g. if $A_i = B_i$ and we give $A_1 \odot A_2$ the maximal norm and $B_1 \odot B_2$ the minimal norm).

We already saw in Corollary 12.18 that this holds when we consider both $A_1 \odot A_2$ and $B_1 \odot B_2$ with their respective minimal tensor product norms.

Exercise 12.49. Show that for a pair of $*$-homomorphisms $\phi_i : A_i \to B_i$, the algebraic tensor product $\phi_1 \odot \phi_2$ extends to a $*$-homomorphism on

$$\phi_1 \otimes_{\max,\beta} \phi_2 : A_1 \otimes_{\max} A_2 \to B_1 \otimes_\beta B_2$$

for any C*-tensor product $B_1 \otimes_\beta B_2$.

However, many maps that we want to work with (e.g. states) are not necessarily $*$-homomorphisms. Hence it is important to understand which class of bounded linear maps extend to tensor products, in particular, for which bounded linear maps $\phi_i : A_i \to B_i$ does $\phi_1 \odot \phi_2$ extend to continuous linear maps

$$\phi_1 \otimes_{\max} \phi_2 : A_1 \otimes_{\max} A_2 \to B_1 \otimes_{\max} B_2$$

and

$$\phi_1 \otimes_{\min} \phi_2 : A_1 \otimes_{\min} A_2 \to B_1 \otimes_{\min} B_2?$$

Let us consider an example where this fails.

Example 12.50. Consider $\mathcal{K} = \mathcal{K}(\ell^2)$. As we saw in Example 12.42, $\mathcal{K}$ is nuclear, meaning in particular that the completion of $\mathcal{K} \odot \mathcal{K}$ under any tensor norm can be identified with the completion of $\mathcal{K} \odot \mathcal{K}$ with respect to the norm on $B(\ell^2 \otimes \ell^2)$ (via the tensor product of faithful representations $id_{\mathcal{K}} \odot id_{\mathcal{K}}$). For each i, j, we define the rank one operator $P_{i,j} = \langle \cdot, e_j \rangle e_i$. (Think of these as an infinite-dimensional version of the matrix units for $\mathrm{M}_n(\mathbb{C})$.) For each $n \in \mathbb{N}$, define $V_n \in \mathcal{K} \otimes \mathcal{K}$ by

$$V_n := \sum_{i,j=1}^{n} P_{i,j} \otimes P_{j,i}.$$

Then V_n is a partial isometry. (Indeed, since $P_{i,j}P_{l,k} = \delta_{j,l}P_{i,k}$, we can compute that $V_n^* V_n = P_n \odot P_n$ where P_n is the rank n projection sending $e_j \mapsto e_j$ for $1 \le j \le n$ and $e_j \mapsto 0$ for $j > n$.) So $\|V_n\| = 1$ for all n.

Now considering each $T = [t_{ij}] \in \mathcal{K}$ as an array, we let $Tr : \mathcal{K} \to \mathcal{K}$ denote the transpose map, which is given by $Tr([t_{ij}]) = [t_{ji}]$. This is a linear $*$-preserving isometric map (since $T^* = [\bar{t}_{ji}]$), and

$$Tr \odot 1_{\mathcal{K}}(V_n) = \sum_{i,j=1}^{n} e_{ji} \otimes e_{ji}.$$

Now, consider the vector $\xi = \sum_{k=1}^{n} e_k \otimes e_k$. One computes

$$\begin{aligned}\|Tr \odot 1_{\mathcal{K}}(V_n)\xi\| &= \|\sum_{i,j=1}^{n} \sum_{k=1}^{n} \langle e_k, e_j \rangle e_i \otimes \langle e_k, e_j \rangle e_i\| \\ &= \|\sum_{i=1}^{n} \sum_{k=1}^{n} \langle e_k, e_k \rangle e_i \otimes \langle e_k, e_k \rangle e_i\| \\ &= \|\sum_{i=1}^{n} n(e_i \otimes e_i)\| = \|n\xi\| = n\|\xi\|.\end{aligned}$$

In particular, this means that $\|Tr \odot 1_{\mathcal{K}}(V_n)\| \ge n$ and hence $\|Tr \odot 1_{\mathcal{K}}\| \ge n$ for all $n \in \mathbb{N}$. This is an unbounded operator and hence not continuous.

So what kinds of bounded linear maps on C*-algebras yield continuous tensor product maps? Notice that the above example is $*$-preserving, so that's not enough. We have remarked several times that much of the structure of the C*-algebra is preserved by positive elements. Perhaps we need to consider linear maps $\phi : A \to B$ that send positive elements in A to positive elements in B? But even that isn't enough. It turns out that the transpose map above does send positive elements to positive elements. So, what gives? This is where we finally motivate the idea of *completely* positive maps. Recall that a linear map $\phi : A \to B$ between C*-algebras is completely positive if (equivalently) the linear map

$$\phi^{(n)} : \mathrm{M}_n(\mathbb{C}) \otimes A \to \mathrm{M}_n(\mathbb{C}) \otimes B$$

is positive for all $n \geq \mathbb{N}$.

Theorem 12.51. *Let $\phi_i : A_i \to B_i$ be linear cp maps. Then the algebraic tensor product map*

$$\phi_1 \odot \phi_2 : A_1 \odot A_2 \to B_1 \odot B_2$$

extends to a linear cp map (which is then also bounded and hence continuous) map on both the maximal and minimal tensor products:

$$\begin{aligned} \phi_1 \otimes \phi_2 :& A_1 \otimes A_2 \to B_1 \otimes B_2 \\ \phi_1 \otimes_{\max} \phi_2 :& A_1 \otimes_{\max} A_2 \to B_1 \otimes_{\max} B_2. \end{aligned}$$

Moreover, we have $\|\phi_1 \otimes_{\max} \phi_2\| = \|\phi_1 \otimes \phi_2\| = \|\phi_1\|\|\phi_2\|$.

Remember that we have already proved this for $*$-homomorphisms. Stinespring's Dilation theorem will allow us to transfer this fact to cp maps.

In full disclosure, we need a stronger version of this to prove the $\otimes_{\max}$ part of Theorem 12.51, so we direct you to [5, Proposition 1.5.6] and its use in the proof of [5, Theorem 3.5.3]. But for the sake of seeing Stinespring's Theorem in action, let's prove that the algebraic tensor product of cp maps extends to a cp map between spatial tensor products.

Proof of Theorem 12.51 (for spatial tensor). Let A_1, A_2, B_1, B_2 be C*-algebras and $\phi_i : A_i \to B_i$ cp maps. First, by taking faithful representations, it suffices to assume that $B_i \subset B(\mathcal{H}_i)$ for $i = 1, 2$ (why?). Then $\phi_i : A_i \to B(\mathcal{H}_i)$ are cp maps, which have Stinespring dilations $(\pi_i, \mathcal{H}_i', V_i)$ for $i = 1, 2$. Since these are $*$-homomorphisms, $\pi_1 \odot \pi_2 : A_1 \odot A_2 \to B(\mathcal{H}_1') \odot B(\mathcal{H}_2') \subset B(\mathcal{H}_1' \otimes \mathcal{H}_2')$ extends to $A_1 \otimes A_2$. Define the map $\phi_1 \otimes \phi_2 : A_1 \otimes A_2 \to B_1 \otimes B_2 \subset B(\mathcal{H}_1' \otimes \mathcal{H}_2')$ by

$$\phi_1 \otimes \phi_2(x) = (V_1 \otimes V_2)^*(\pi_1 \otimes \pi_2)(x)(V_1 \otimes V_2).$$

By Example 10.11, this is a cp map. Moreover, for elementary tensors $a_1 \odot a_2 \in A_2 \odot A_2$, we have

$$\phi_1 \otimes \phi_2(a_1 \odot a_2) = (V_1^* \pi_1(a_1) V_1) \otimes (V_2^* \pi_2(a_2) V_2) = \phi_1(a_1) \odot \phi_2(a_2),$$

which means (by linearity) that $\phi_1 \otimes \phi_2|_{A_1 \odot A_2} = \phi_1 \odot \phi_2$. □

13. Amenability

Preview of Lecture: In lecture, we'll discuss the paradoxical decomposition of $\mathbb{F}_2$ (Example 13.4), but probably not the proof of Proposition 13.5 or Proposition 13.6. My goal in lecture will be to discuss the proof of Theorem 13.17; this will require also discussing Følner sets, but we won't get into the proof of Proposition 13.13 or Proposition 13.16.

The concept of amenability for groups was introduced by John von Neumann in 1929[32], in response to the Banach–Tarski paradox. For modern operator algebraists, amenable groups are important because these are precisely the groups G for which $C^*(G) \cong C^*_r(G)$. Another C*-algebraic characterization of amenability is that G is amenable iff $C^*_r(G)$ is nuclear – indeed, this is what underlies the use of the word "amenable" instead of "nuclear" for more general C*-algebras. More generally, if a C*-algebra A is nuclear and $\alpha : G \to \mathrm{Aut}(A)$ is an action of an amenable group on A, then the crossed product C*-algebra $C^*(G, A, \alpha)$ will be nuclear.

There are many (many) equivalent characterizations of amenability (and they all have analogues for locally compact groups, although in these notes we'll just treat the discrete case). If you want to know more than what's presented here, [5, Section 2.6] is a good place to start. For a more exhaustive account, check out [13] or [7, Chapter 4].

Definition 13.1. A discrete group G is *amenable* if it admits a left-invariant mean: that is, there is a state[33] (a.k.a. mean) μ on $\ell^\infty(G)$ such that

$$\mu(f) = \mu(g \mapsto f(sg)) =: \mu(\lambda_{s^{-1}}(f))$$

for all $f \in \ell^\infty$ and $s \in G$. (By abuse of notation, we use the same symbol $\lambda_{s^{-1}}$ for the left-translation action of G on $\ell^\infty(G)$ that we used for the left action of G on $\ell^2(G)$.)

Example 13.2. Any finite group G is amenable. We define $\mu(\delta_g) = \frac{1}{|G|}$ for each $g \in G$. It is easy to check that if we extend μ to $\ell^\infty(G)$ by requiring it to be linear, the result is a state.

Remark 13.3. Two nice facts about means on groups:

(1) For a discrete group G, a finitely additive probability measure is a function $m : 2^G \to [0, 1]$ which satisfies $m(G) = 1$ and $m(S \sqcup T) = m(S) + m(T)$ for all disjoint $S, T \subset G$. There is a G-invariant bijective correspondence between means on $\ell^\infty(G)$ and finitely additive probability measures on 2^G given by $\mu \mapsto m_\mu$ where
$$m_\mu(A) = \mu(\chi_A),\ \forall A \subset G.$$
For a proof, check out [7, Theorem 4.1.8 and Remark 4.3.5]. The fact that the map is G-invariant means that G admits a left-invariant mean iff it admits a left invariant finitely additive probability measure.

(2) The set of means on G forms a convex weak*-compact subset of $(\ell^\infty(G))^*$.

Example 13.4. The free group $\mathbb{F}_2$ is not amenable.

Recall that $\mathbb{F}_2 = \langle a, b\rangle$ is the set of all words in two noncommuting generators (here called a, b) and their inverses. We will assume that the words are *reduced* in the sense that a variable is never immediately followed by its inverse.

Let A_+ denote the set of words in $\mathbb{F}_2$ whose first letter is a, and A_- denote the set of words whose first letter is a^{-1}; likewise with B_+ and B_-. Since every reduced word (that is not the identity element) starts with a, b, a^{-1} or b^{-1}, we have

$$\mathbb{F}_2 = A_+ \sqcup A_- \sqcup B_+ \sqcup B_- \sqcup \{e\}$$

Since the words in A_- are reduced, the second "letter" of any word in A_- cannot be a; likewise for B_- and b. Hence we have

$$A_+ \sqcup aA_- = \mathbb{F}_2 = B_+ \sqcup bB_-.$$

[32] The original word coined by von Neumann was "messbar," but but his co-author Mahlon Day, gave "amenable" as the English translation ... because it's a pun ... a-MEAN-able... get it?

[33] We've only defined states on C*-algebras so far, but the definition in this context is the same: a linear functional of norm 1 which assigns a nonnegative real number to any nonnegative function.

Now suppose $\mu \in \ell^\infty(\mathbb{F}_2)$ is a left-invariant mean and $m_\mu : 2^{\mathbb{F}_2} \to [0,1]$ is the corresponding left invariant finitely additive probability measure from Remark 13.3. Then we have

$$\begin{aligned} 1 = m_\mu(\mathbb{F}_2) &= m_\mu(A_+) + m_\mu(A_-) + m_\mu(B_+) + m_\mu(B_-) + m_\mu(\{e\}) \\ &= m_\mu(A_+) + m_\mu(aA_-) + m_\mu(B_+) + m_\mu(bB_-) + m_\mu(\{e\}) \\ &= m_\mu(A_+ \sqcup aA_-) + m_\mu(B_+ \sqcup bB_-) + m_\mu(\{e\}) \\ &= 2m_\mu(\mathbb{F}_2) + m_\mu(\{e\}) \geq 2. \end{aligned}$$

The issue is the existence of disjoint subsets $A_+ \sqcup A_-$ and $B_+ \sqcup B_-$ of $\mathbb{F}_2$, which both end up having the same measure as $\mathbb{F}_2$ under any translation-invariant measure. This means $\mathbb{F}_2$ admits a *paradoxical decomposition*, and this is what underlies the Banach-Tarski paradox. It was shown by von Neumann[34] that a group is amenable iff it does *not* admit a paradoxical decomposition.

Proposition 13.5. *If G is abelian then G is amenable.*

The proof uses the *Markov-Kakutani fixed point theorem* [8, Theorem VII.2.1]: if X is a topological vector space, $K \subseteq X$ is compact and convex, and T is a collection of continuous, linear, pairwise commuting maps $t : X \to X$ is such that every $t \in T$ satisfies $tK \subseteq K$, then there is a point in K which is fixed by all $t \in T$.

Proof of Proposition 13.5. The compact convex set K of interest here is the set $\mathcal{S}(\ell^\infty(G))$ of states ϕ on $\ell^\infty(G)$; take $T = \{\lambda_s^* : s \in G\}$, where

$$\lambda_s^*(\phi)(f) = \phi(\lambda_s f) = \phi(g \mapsto f(s^{-1}g)).$$

Then one checks that every element of T is continuous, in the sense that if a net $(\phi_i)_i \in \ell^\infty(G)^*$ satisfies $\phi_i \to \phi$ in the weak-* topology, then $\lambda_s^*(\phi_i) \to \lambda_s^*(\phi)$ for all $s \in G$. The fact that G is abelian implies that T is a set of pairwise commuting maps, and one can check that T preserves $\mathcal{S}(\ell^\infty(G))$. So, the Markov-Kakutani fixed point theorem gives us $\mu \in \mathcal{S}(\ell^\infty(G))$ such that $\lambda_s^*(\mu) = \mu$ for all s. By construction, μ is a left-invariant mean on $\ell^\infty(G)$. □

Proposition 13.6. *The class of amenable groups is closed under taking subgroups, quotients, extensions, and inductive limits.*

Proof. We will prove that the class of amenable groups is closed under extensions, and leave the rest as an exercise. So, suppose that N, H are amenable, with left invariant means μ_N, μ_H respectively, and $1 \to N \to G \to H \to 1$ is a short exact sequence of groups (so that N is normal in G and $H \cong G/N$). We define a functional μ on $\ell^\infty(G)$ by

$$\mu(f) = \mu_H(sN \mapsto \mu_N(g \mapsto f(sg))).$$

(In the above equation, $g \in N$ and $s \in G$.) Notice that the function $sN \mapsto \mu_N(g \mapsto f(sg))$ is well defined by our hypothesis that μ_N is left invariant; for any $n \in N$ we have

$$\mu_N(g \mapsto f(sg)) = \mu_N(g \mapsto f(sng)).$$

Since μ_H and μ_N have norm 1, so will μ. (**Exercise:** Convince yourself of this!) Moreover, if f is positive, then the fact that μ_H, μ_N are positive linear functionals implies that μ is also a positive linear functional. To see that μ is indeed a left invariant mean, then, it merely remains to check left invariance. If $\tilde{g} \in G$, then

$$\mu(\lambda_{\tilde{g}} f) = \mu_H(sN \mapsto \mu_N(g \mapsto (\lambda_{\tilde{g}} f(sg))) = \mu_H(sN \mapsto \mu_N(g \mapsto f(\tilde{g}^{-1}sg))) = \mu_H(\tilde{g}^{-1}sN \mapsto \mu_N(g \mapsto f(\tilde{g}^{-1}sg)))$$

by the left invariance of μ_H. However, replacing $s \in G$ with $\tilde{g}s$ reveals that this final quantity is precisely $\mu(f)$, as desired. □

Exercise 13.7. Complete the proof of Proposition 13.6. Some hints:

- If $H \leq G$ is a subgroup of an amenable group, pick a set S of left coset representatives of $H \leq G$, so that you can write any $g \in G$ uniquely as $g = sh$ for $s \in S, h \in H$. Use this to embed $\ell^\infty(H)$ into $\ell^\infty(G)$.
  (Side question: Why can't we just define μ by $\frac{\mu|_H}{\mu(H)}$?)
- To show that $G = \varinjlim G_n$ is amenable whenever all the groups G_n are, you'll need to take a weak-$*$ cluster point of the left invariant means witnessing amenability of the G_ns.

[34] Starting a sentence with "von Neumann" is arguably against the rules of grammar.

In particular, Proposition 13.6 implies that $\mathbb{F}_n$ is not amenable for any $n \geq 2$: Each such $\mathbb{F}_n$ contains $\mathbb{F}_2$ as a subgroup.

Theorem 13.8. *G is amenable iff* $\mathrm{C}^*_r(G) \cong \mathrm{C}^*(G)$.

Proof. We will prove the backwards direction; the forwards direction (cf. [8, Theorem VII.2.8] or [5, Theorem 2.6.8]) uses a lot of machinery that we don't have time to introduce.

Suppose $\mathrm{C}^*_r(G) \cong \mathrm{C}^*(G)$. Note that the universal property of $\mathrm{C}^*(G)$ means that it always admits a one-dimensional representation χ, given by $\chi(u_g) = 1 \in \mathbb{C}$ for all $g \in G$. Then, since we assumed that the canonical surjection $\pi_\lambda : \mathrm{C}^*(G) \to \mathrm{C}^*_r(G)$ is an isomorphism, χ becomes a 1-dimensional representation on $\mathrm{C}^*_r(G) \subseteq B(\ell^2(G))$. Arveson's extension theorem then tells us that χ extends to a state on $B(\ell^2(G))$, which then restricts to a state on $\ell^\infty(G)$.

It's straightforward to check **Exercise: do it!** that if $f \in \ell^\infty(G)$, $f = \sum_{g\in G} a_g \delta_g$, then $\lambda_s(f) = u_s f u_s^*$ as operators on $\ell^2(G)$. Moreover, as χ is a $*$-homomorphism on $\mathrm{C}^*_r(G) \ni u_g$, we know that all the generators u_g lie in the multiplicative domain of χ (see Proposition 10.28). Therefore, $\chi(\lambda_s f) = \chi(f)$ for any $f \in \ell^\infty(G)$, so χ is our left-invariant mean. □

We would also like to prove that G is amenable iff $\mathrm{C}^*_r(G)$ is nuclear. To do this, it will be easier to work with a different characterization of amenability. To introduce it, recall that if S, T are sets, then $S\Delta T = (S \cup T)\backslash(S \cap T)$ is the set of elements which are in precisely one of S, T.

Definition 13.9. A discrete group G satisfies the *Følner condition* if for any finite subset $E \subseteq G$ and any $\epsilon > 0$, there is a finite subset $F \subseteq G$ such that

$$\frac{|sF\Delta F|}{|F|} < \epsilon \text{ for all } s \in E.$$

Example 13.10. For $G = \mathbb{Z}^d$, the usual choice of Følner sets (the sets F in the definition above) are of the form $F_N = \{0, ..., N\}^d$ (sometimes written as $[0, N]^d$).

Exercise 13.11. Check that $\mathbb{Z}$ satisfies the Følner condition.

It is a fact (see Section 13.1) that G satisfies the Følner condition iff G is amenable. As this takes a while to prove, we will prove here that satisfying the Følner condition is equivalent to the following property, which is hopefully sufficiently reminiscent of the definition of amenability that you're willing to believe said fact. If you recall that $\ell^1(G)$ is the predual of $\ell^\infty(G)$ and hence is dense in $\ell^\infty(G)^*$, you may be even more credulous.

Definition 13.12. A discrete group G admits an *approximate invariant mean*[35] if, for any finite subset $E \subseteq G$ and any $\epsilon > 0$, there is a positive function $m = m(E, \epsilon) \in \ell^1(G)$ with $\sum_{s\in G} m(s) = 1$ and such that

$$\sup_{s\in E} \sum_{t\in G} |m(s^{-1}t) - m(t)| < \epsilon.$$

This is also sometimes referred to as *Reiter's condition.*

Proposition 13.13. *G satisfies the Følner condition iff G admits an approximate invariant mean.*

Proof. Suppose G satisfies the Følner condition. Given a finite set E and $\epsilon > 0$, let $F \subseteq G$ be the finite set guaranteed by the Følner condition and let $m = \frac{1}{|F|}\chi_F$. Note that

$$\chi_F(s^{-1}t) = 1 \Leftrightarrow s^{-1}t \in F \Leftrightarrow t \in sF,$$

so $\sum_{t\in G} |m(s^{-1}t) - m(t)| = \frac{|sF\Delta F|}{|F|} < \epsilon$ for all $s \in E$.

On the other hand, suppose that G admits an approximate invariant mean. We first make a helpful technical observation. Given a positive function $f \in \ell^1(G)$ and $r \geq 0$, set $F(f, r) = \{t : f(t) > r\}$. Notice

[35]Using the canonical identification of $\ell^1(G) \subset \ell^1(G)^{**} = \ell^\infty(G)^*$, we can see that any such m is a mean on G. In fact, one can show using a Hahn-Banach argument that functions m satisfying Reiter's condition are weak*-dense in the (convex weak*-compact) subset $M(G) \subset \ell^\infty(G)^*$ of all means on G.

first that $F(f,r)$ must be finite for each fixed r, in order to have $f \in \ell^1(G)$. We now observe that if f, h are two such functions, both bounded above by 1, then

$$|f(t) - h(t)| = \int_0^1 |\chi_{F(f,r)}(t) - \chi_{F(h,r)}(t)|\, dr.$$

To see this, suppose without loss of generality that $f(t) = x, h(t) = y$ with $x \leq y$. Then $\chi_{F(f,r)}(t) = 1$ iff $r < x$ and $\chi_{F(h,r)}(t) = 1$ iff $r < y$, so the integrand is 1 precisely on the interval $[x, y)$.

Now, supposing G admits an approximate invariant mean, fix a finite subset $E \subseteq G$ and $\delta > 0$; write $\epsilon = \delta/|E|$, and let $m \in \ell^1(G)$ be a norm-1 positive function such that $\sum_{t\in G} |m(t) - m(s^{-1}t)| < \epsilon$ for all $s \in E$. Applying our above observation to the functions $f = m, h = \lambda_s m$, we have

$$\sum_{t\in G} |m(t) - m(s^{-1}t)| = \sum_{t\in G} \int_0^1 |\chi_{F(f,r)}(t) - \chi_{F(h,r)}(t)|\, dr = \int_0^1 \sum_{t\in G} |\chi_{F(f,r)}(t) - \chi_{F(h,r)}(t)|\, dr$$

(as the integrand is positive we can exchange the integral and the sum). Moreover, we have $t \in F(h,r)$ precisely if $m(s^{-1}t) > r$, that is, if $t \in sF(m,r)$. It follows that

$$\sum_{t\in G} |m(t) - m(s^{-1}t)| = \int_0^1 |F(m,r)\,\Delta\, sF(m,r)|\, dr < \epsilon$$

for all $s \in G$. Furthermore, as m has ℓ^1 norm 1, $1 = \sum_{t\in G} m(t) = \int_0^1 |F(m,r)|\, dr$. It follows that

$$\sum_{s\in E} \int_0^1 |sF(m,r)\,\Delta\, F(m,r)|\, dr < \int_0^1 |E|\epsilon |F(m,r)|\, dr,$$

and so we must have

$$\sum_{s\in E} |sF(m,r)\,\Delta\, F(m,r)| < |E|\epsilon|F(m,r)|$$

for some r. Then, in particular, for each $s \in E$ we have

$$\frac{|sF(m,r)\,\Delta\, F(m,r)|}{|F(m,r)|} < |E|\epsilon = \delta,$$

so $F(m,r)$ satisfies the Følner condition for the given E and $\delta > 0$. □

The proof of the following Proposition can be found in Section 13.1. It uses a lot more Banach space theory than one might expect.

Proposition 13.14. *G is amenable iff G admits an approximate invariant mean (iff G satisfies the Følner condition).*

Remark 13.15. Using Propositions 13.6 and 13.14, we can generate a new proof that all abelian groups are amenable. First, we remark that any group is the inductive limit of its finitely generated subgroups (where the connecting maps are just inclusions). Hence, it suffices to prove that every finitely generated abelian group is amenable. The Fundamental Theorem of Finitely Generated Abelian Groups says that any such group is isomorphic to one of the form $\mathbb{Z}^d \times \mathbb{Z}/(p_1^{n_1}\mathbb{Z}) \times \ldots \times \mathbb{Z}/(p_1^{n_m}\mathbb{Z})$ for some $d, m \geq 0$, primes $p_1, ..., p_m$ and $n_1, ..., n_m \geq 0$. This group can be realized as the following extension:

$$0 \longrightarrow \mathbb{Z}^d \longrightarrow \left(\mathbb{Z}^d \times \frac{\mathbb{Z}}{p_1^{n_1}\mathbb{Z}} \times \ldots \times \frac{\mathbb{Z}}{p_1^{n_m}\mathbb{Z}}\right) \longrightarrow \left(\frac{\mathbb{Z}}{p_1^{n_1}\mathbb{Z}} \times \ldots \times \frac{\mathbb{Z}}{p_1^{n_m}\mathbb{Z}}\right) \longrightarrow 0.$$

Since the quotient is finite, all we have to show is that $\mathbb{Z}^d$ is amenable. Oh, we already did that with Følner sets!

Before proving our next theorem, we need the following useful fact about completely positive maps (due to Choi). In what follows, $E_{ij} \in \mathrm{M}_n(C)$ is the "matrix unit" whose entries are all 0 except for the i, j^{th} entry, which is 1.

Proposition 13.16. *A map $\phi : \mathrm{M}_n(\mathbb{C}) \to A$ is completely positive iff $[\phi(E_{ij})]_{i,j} \in \mathrm{M}_n(A)$ is positive.*

Proof. We prove the backwards direction and leave the forwards direction as an easy **Exercise** to the reader. So, suppose $a = [\phi(E_{ij})]_{i,j} \in \mathrm{M}_n(A)$ is positive; write $[b_{ij}]_{i,j} := a^{1/2}$, so that

$$a_{ij} = \phi(E_{ij}) = (b^*b)_{ij} = \sum_{k=1}^n b_{ki}^* b_{kj}.$$

Without loss of generality, assume $A \subseteq B(\mathcal{H})$, so that each entry b_{ij} of $b \in \mathrm{M}_n(A)$ lies in $B(\mathcal{H})$. Define $V : \mathcal{H} \to \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathcal{H}$ by

$$V(\xi) = \sum_{j,k=1}^n e_j \otimes e_k \otimes b_{k,j}\xi,$$

where $\{e_j\}_{j=1}^n$ are the canonical basis vectors for $\mathbb{C}^n$. Then we compute that for $T = [t_{ij}] \in \mathrm{M}_n(\mathbb{C})$,

$$\begin{aligned}\langle V^*(T \otimes 1 \otimes 1)V\eta, \xi\rangle &= \langle (T \otimes 1 \otimes 1)(V\eta), V\xi\rangle \\ &= \langle \sum_{i,j,k=1}^n t_{ij} e_i \otimes e_k \otimes b_{k,j}\eta, \sum_{\ell,m=1}^n e_\ell \otimes e_m \otimes b_{m,\ell}\xi\rangle \\ &= \sum_{i,j,k=1}^n t_{ij}\langle b_{k,j}\eta, b_{k,i}\xi\rangle = \sum_{i,j,k=1}^n t_{ij}\langle b_{k,i}^* b_{k,j}\eta, \xi\rangle \\ &= \langle \phi([t_{ij}])\eta, \xi\rangle.\end{aligned}$$

In other words, $\phi(T) = V^*(T\otimes 1\otimes 1)V$ is a compression of the $*$-homomorphism $\psi : \mathrm{M}_n(\mathbb{C}) \to B(\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathcal{H})$ given by $\psi(T) = T \otimes 1 \otimes 1$, so (as we saw in Exercise 10.10) ϕ is cp. □

Finally, we can prove our second marquee theorem.

Theorem 13.17. *G is amenable iff $\mathrm{C}_r^*(G)$ is nuclear.*

Proof. Suppose G is amenable (and, for simplicity, countable, so that we can enumerate the elements of G). By Proposition 13.14, we can assume that G satisfies the Følner condition. Choose, then, a sequence of finite sets F_n such that F_n satisfies the Følner condition for $\epsilon = 1/n$ and the finite set consisting of the first n elements of G. Let $P_n \in B(\ell^2(G))$ be the projection onto the subspace spanned by $\{\delta_g : g \in F_n\}$, so that we can identify $P_n B(\ell^2(G))P_n$ with $M_{F_n}(\mathbb{C})$. Define $\phi_n : \mathrm{C}_r^*(G) \to M_{F_n}(\mathbb{C})$ by $\phi_n(x) = P_n x P_n$. Example 10.11 shows that ϕ_n is ccp.

To define $\psi_n : M_{F_n}(\mathbb{C}) \to \mathrm{C}_r^*(G)$, write E_{pq} for the matrix unit in $M_{F_n}(\mathbb{C})$ such that $E_{pq}(\delta_q) = \delta_p$. Then define

$$\psi_n(E_{pq}) = \frac{1}{|F_n|} u_p u_q^*,$$

and extend ψ_n to be a linear map on $M_{F_n}(\mathbb{C})$. If we enumerate the elements of F_n as $p_1, \ldots, p_{|F_n|}$, then $[\psi_n(E_{pq})]$ satisfies

$$[\psi_n(E_{pq})] = \frac{1}{|F_n|}\begin{bmatrix} u_{p_1} & 0 & \cdots & 0 \\ u_{p_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ u_{p_{|F_n|}} & 0 & \cdots & 0\end{bmatrix}\begin{bmatrix} u_{p_1} & 0 & \cdots & 0 \\ u_{p_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ u_{p_{|F_n|}} & 0 & \cdots & 0\end{bmatrix}^* \geq 0,$$

so Proposition 13.16 tells us that ψ_n is also cp. In fact, ψ is ucp: our choice of scaling factor and the fact that each u_p is a unitary means that

$$\psi_n(1) = \sum_{p \in F_n} \psi_n(E_{pp}) = 1.$$

To complete the proof that $\mathrm{C}_r^*(G)$ is nuclear when G is amenable, it remains to show that for any $a \in \mathrm{C}_r^*(G)$ we have $\lim_{n\to\infty} \|a - \psi_n(\phi_n(a))\| = 0$. In fact, since the generators u_s densely span $\mathrm{C}_r^*(G)$, it suffices to show that $\lim_{n\to\infty} \|u_s - \psi_n(\phi_n(u_s))\| = 0$ for all $s \in G$.

One quickly computes that $\phi_n(u_s) = \sum_{p : p, s^{-1}p \in F_n} E_{p, s^{-1}p}$, and therefore

$$\psi_n(\phi_n(u_s)) = \frac{1}{|F_n|}\sum_{p:p,s^{-1}p\in F_n} u_p u_{s^{-1}p}^* = \frac{1}{|F_n|}\sum_{p:p,s^{-1}p\in F_n} u_s = u_s \frac{|F_n \cap sF_n|}{|F_n|}.$$

As $|F_n \,\Delta\, sF_n| = 2|F_n| - 2|F_n \cap sF_n|$, our choice of the sets F_n implies that

$$0 = \lim_{n\to\infty} \frac{|F_n \,\Delta\, sF_n|}{|F_n|} = \lim_{n\to\infty} 1 - \frac{|F_n \cap sF_n|}{|F_n|}$$

for any $s \in G$. In particular,

$$\lim_{n\to\infty} \|u_s - \psi_n(\phi_n(u_s))\| = \lim_{n\to\infty} 1 - \frac{|F_n \cap sF_n|}{|F_n|} = 0,$$

as desired.

Now, for the converse. Assume $C^*_r(G)$ is nuclear, so that we have cpc maps $\phi_n : C^*_r(G) \to M_{k(n)}$ and $\psi_n : M_{k(n)} \to C^*_r(G)$. By Arveson's Extension Theorem, we might as well assume that ϕ_n is defined on all of $B(\ell^2(G))$, so that the composition $\Phi_n = \psi_n \circ \phi_n$ is a cpc map from $B(\ell^2(G))$ to $C^*_r(G)$, such that $\Phi_n(x) \to x$ for all $x \in C^*_r(G)$. Take a point-ultraweak limit of the maps Φ_n (ask Brent and Rolando), and we end up with a cpc map $\Phi : B(\ell^2(G)) \to L(G)$ which restricts to the identity on $C^*_r(G)$.

Recall from your von Neumann algebra lectures that there is a canonical trace τ on $L(G)$, given by $\tau(x) = \langle x\delta_e, \delta_e\rangle$. Define $\mu = \tau \circ \Phi$; we claim that μ is a left invariant mean. To see this, we again use that the left translation action λ_s on functions in $\ell^\infty(G) \subseteq B(\ell^2(G))$ is given by $\lambda_s(f) = u_s f u_s^*$. Since $\Phi|_{C^*_r(G)} = \mathrm{id}$, we have u_g in the multiplicative domain of Φ for all g. Consequently, for any $f \in \ell^\infty(G)$,

$$\mu(\lambda_s(f)) = \tau(\Phi(u_s f u_s^*)) = \tau(u_s \Phi(f) u_s^*) = \tau(\Phi(f)),$$

since τ is a trace and u_s is a unitary. □

13.1. **Amenability ⇔ Følner Condition.** One can find the following proof (via some extra steps) in [5, Theorem 2.6.8] (see also [8, Theorem VII.2.8]). These use some more heavy duty Banach space theory. Below is a proof coming partially from [7, Section 4.9] and partly from Terry Tao's Blog, which cleverly circumvents some of the difficulty by considering approximate invariant *finite* means.

Definition 13.18. A *finite* mean is a positive function $f \in \ell^1(G)$ with finite support (i.e., $f(s) = 0$ for all but finitely many $s \in G$) and $\|f\|_1 = \sum_{s\in G} f(s) = 1$.

Why are these called means? Recall that $\ell^\infty(G) \simeq \ell^1(G)^*$ and $\ell^1(G)$ embeds isometrically[36] into $\ell^1(G)^{**} \subset \ell^\infty(G)^*$. In particular, a function $g \in \ell^\infty(G)$ acts as a linear functional on $\ell^1(G)$ via

$$f \mapsto \sum_{s\in G} f(s)g(s), \ \forall\, f \in \ell^1(G)$$

and likewise, a function $f \in \ell^1(G)$ acts as a linear functional on $\ell^\infty(G)$ via

$$g \mapsto \sum_{s\in G} f(s)g(s), \ \forall\, g \in \ell^\infty(G).$$

With this in mind, we can view the finite means as means (i.e., states on $\ell^\infty(G)$). Moreover, these are exactly the means that arise as ℓ^1-functions of finite support; in symbols

$$\mathrm{Prob}_{\mathrm{fin}}(G) := \{\text{finite means on } G\} = \mathcal{S}(\ell^\infty(G)) \cap \{f \in \ell^1(G) \mid |\mathrm{supp}(f)| < \infty\}.$$

In fact, one can show using a Hahn-Banach argument that these are weak*-dense in $\mathcal{S}(\ell^\infty(G))$.

Remark 13.19. Notice that regardless of whether $f \in \ell^1(G)$ is acting as linear functional on $g \in \ell^\infty(G)$ or vice versa, we get the same output, namely $\sum_{s\in G} f(s)g(s)$. For this reason, we sometimes use what is called "bra-ket" notation.[37] We simply define the function

$$\langle \cdot, \cdot \rangle : \ell^1(G) \times \ell^\infty(G) \to \mathbb{R}$$

by $\langle f, g\rangle = \sum_{s\in G} f(s)g(s)$ for all $(f, g) \in \ell^1(G) \times \ell^\infty(G)$. Since this can take a little getting used to, we will just write $\hat{f}$ (resp. $\widehat{G}$) when we are thinking of $f \in \ell^1(G)$ (resp. $g \in \ell^\infty(G)$) as a functional.

[36] meaning that for $f \in \ell^1(G)$, we have $\|f\|_1 = \sum_{s\in G} |f(s)|$, which agrees with its norm when we view it as a linear functional on $\ell^\infty(G)$

[37] I did not make this up. I blame physicists.

Example 13.20. For $s \in G$, consider the Kronecker delta function $\delta_s : G \to \mathbb{R}$ given by $\delta_s(t) = 1$ for $t = s$ and 0 otherwise. Then for each $s \in S$, $\delta_s \in \ell^1(G)$ has finite support (just s) with $\|\delta_s\|_1 = \delta_s(s) = 1$, and for any $g \in \ell^\infty(G)$, we have

$$\langle \delta_s, g\rangle = \hat{\delta}_s(g) = \sum_{t\in G} \delta_s(t)g(t) = g(s).$$

In other words, $\hat{\delta}_s$ is the evaluation function at s for $g \in \ell^\infty(G)$.

Theorem 13.21. *Let G be a discrete group. Then the following are equivalent.*

(1) G is amenable.
(2) For any finite subset $E \subset G$ and $\varepsilon > 0$, there exists a finite mean f so that

$$\|f - s.f\|_1 < \varepsilon.$$

(3) G satisfies the Følner condition.

Proof. We prove (1) ⇒ (2) ⇒ (3) ⇒ (1).

Suppose (2) fails. Then there exists a finite subset $E = \{s_1, ..., s_n\} \subset G$ and $\varepsilon > 0$ so that for all $f \in \mathrm{Prob}_{\mathrm{fin}}(G)$,

$$\max_{s\in E} \|f - s.f\|_1 \geq \varepsilon.$$

If we equip $\ell^1(G)^n$ with the norm $\|(f_1, ..., f_n)\|_\infty = \max_{1\leq j\leq n} \|f_j\|_1$, then we have a Banach space, and the set

$$C := \bigoplus_{i=1}^{n} \big\{(f - s_i \cdot f) \mid f \in \mathrm{Prob}_{\mathrm{fin}}(G)\big\} \subset \ell^1(G)^n$$

is bounded away from 0 in this norm, which means $0 \notin \overline{C}$. Moreover, since $\mathrm{Prob}_{\mathrm{fin}}(G) \subset \ell^1(G)$ is convex (**Exercise: check**), it follows that C is also convex (**Exercise: check**).[38] Hence, we have a closed convex set $\overline{C}$ with $0 \notin \overline{C}$.

Hence the ($\mathbb{R}$-)Hahn Banach separation theorem says there exists a $\Phi \in (\ell^1(G))^n)^*$ and $\varepsilon' \geq 0$ so that

$$\Phi\big((f - s_i \cdot f)_{i=1}^n\big) \geq \varepsilon' > 0 = \Phi(0),\ \forall\ (f - s_i \cdot f)_{i=1}^n \in \overline{C}. \tag{13.1}$$

For each $1 \leq i \leq n$, define $\phi_i \in \ell^1(G)$ by $\phi_i(f) := \Phi(0, ..., f, ...0)$ (where f is appearing in the ith coordinate) for all $f \in \ell^1(G)$. Then $\phi_i \in \ell^1(G)^*$ for each $1 \leq i \leq n$, and moreover for $(f_1, ..., f_n) \in \ell^1(G)^n$, we have

$$(\phi_1, ..., \phi_n)\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \sum_{i=1}^n \phi_i(f_i) = \sum_{i=1}^n \Phi((0, ..., f_i, ...0)) = \Phi\big(\sum_{i=1}^n (0, ..., f_i, ...0)\big) = \Phi((f_1, ..., f_n)).$$

With this, (13.1) (along with linearity and the definition of the induced group action on $\ell^1(G)^*$) tells us that for all $f \in \mathrm{Prob}_{\mathrm{fin}}(G)$,

$$\begin{aligned} 0 < \varepsilon' \leq \Phi\big((f - s_i \cdot f)_{i=1}^n\big) &= \sum_{i=1}^n \phi_i(f - s_i \cdot f) = \sum_{i=1}^n \phi_i(f) - \phi_i(s_i \cdot f) = \sum_{i=1}^n \phi_i(f) - (s_i^{-1} \cdot \phi_i)(f) \\ &= \left(\sum_{i=1}^n \phi_i - (s_i^{-1} \cdot \phi_i)\right)(f). \end{aligned}$$

Using duality, we identify each ϕ_i with $\widehat{G}_i$ for some $g_i \in \ell^\infty(G)$ (where $\widehat{G}_i$ indicates that we are thinking of the $\ell^\infty(G)$ function as a functional on $\ell^1(G)$)[39], and set $g = \sum_{i=1}^n \big(g_i - s_i^{-1} \cdot g_i\big) \in \ell^\infty(G)$. Now, the above estimate tells us that for all $f \in \mathrm{Prob}_{\mathrm{fin}}(G)$,

$$0 < \varepsilon' \leq \sum_{i=1}^n \big(\widehat{G}_i - (s_i^{-1} \cdot \widehat{G}_i)\big)(f) = \widehat{G}(f).$$

[38] Getting convexity is exactly why we are bothering with this direct sum business.

[39] In fact, if we equip $\ell^\infty(G)^n$ with the 1-norm $\|(g_1, ..., g_n)\|_1 = \sum_{i=1}^n \|g_i\|_\infty$ and $\ell^1(G)^n$ with the ∞-norm $\|(f_1, ..., f_n)\|_\infty = \max_{1\leq i\leq n} \|f_i\|_1$, then this gives us an isomorphism of the two spaces.

In particular, this holds for $f = \delta_s$ for any $s \in G$, which (using Example 13.20) gives us

$$0 < \varepsilon' \leq \widehat{G}(\delta_s) = g(s), \ \forall \ s \in G.$$

Hence $g = \sum_{i=1}^n \left(g_i - s_i^{-1} \cdot g_i\right) \in \ell^\infty(G)$ is a function with $g(s) \geq \varepsilon' > 0$ for all $s \in G$.

It follows that for every mean $m \in \mathcal{S}(\ell^\infty(G))$, we have $m(g) \geq \varepsilon' > 0$ (since $m(g - \varepsilon' 1) \geq 0$ where $m(\varepsilon' 1) = \varepsilon' m(1) = \varepsilon'$). However, if $m \in \mathcal{S}(\ell^\infty(G))$ is an invariant mean, then in particular,

$$m(g_i - s_i^{-1} \cdot g_i) = m(g_i) - (s_i \cdot m)(g_i) = (m - s_i \cdot m)(g_i) = 0, \ \forall \ 1 \leq i \leq n.$$

However, that gives

$$0 < m(g) = m\left(\sum_{i=1}^n g_i - s_i^{-1} \cdot g_i\right) = \sum_{i=1}^n m(g_i - s_i^{-1} \cdot g_i) = 0,$$

a contradiction.

For (2) $\Rightarrow$ (3), fix a nonempty finite set $E \subset G$ and $\varepsilon > 0$. Choose $f \in \mathrm{Prob_{fin}}(G)$ so that

$$\max_{s \in E} |f - s \cdot f\|_1 < \frac{\varepsilon}{|E|}.$$

The trick is to write f as a "layer-cake decomposition." This can be done in more generality, but since f only takes on finitely many real positive values, we can do it hands-on with a cute telescoping trick: Since f has finite support, its range consists of only finitely many distinct positive real numbers, which we order as $r_1 < r_2 < ... < r_k$. For each $1 \leq i \leq k$, set $E_k := \{s \in G \mid f(s) \geq r_k\}$ and set $c_1 = r_1$ and $c_j = r_j - r_{j-1}$ for $1 < j \leq k$. Then we have $E_k \subset E_{k-1} \subset ... \subset E_1$, and

$$f = \sum_{j=1}^n c_j E_j.$$

Moreover, by assumption, we have that

$$1 = \|f\|_1 = \sum_{j=1}^k c_j |E_j|.$$

Now, for each $s \in S$ and $1 \leq j \leq k$, and $t \in sE_j \Delta E_j$, we have

$$(f - s \cdot f)(t) = \begin{cases} c_i & t \in E_j \backslash sE_j \\ -c_i & t \in sE_j \backslash E_j \end{cases}$$

Either way, $|f - s \cdot f| \geq c_i$ on $sE_j \Delta E_j$. From this, we have for each $s \in S$,

$$\sum_{j=1}^k c_j |sE_j \Delta E_j| \leq \|f - s \cdot f\|_1 < \frac{\varepsilon}{|E|} =< \frac{\varepsilon}{|E|}(1) =< \frac{\varepsilon}{|E|} \sum_{j=1}^k c_j |E_j|.$$

Hence

$$\sum_{j=1}^k \sum_{s \in E} c_j |sE_j \Delta E_j| = \sum_{s \in E} \sum_{j=1}^k c_j |sE_j \Delta E_j| < |E| \frac{\varepsilon}{|E|} \sum_{j=1}^k c_j |E_j| = \sum_{j=1}^k \varepsilon c_j |E_j|.$$

What we have on the left hand side is the sum of k positive real numbers, which is smaller than the sum of the k positive real number on the right hand side. It follows that for some $1 \leq i \leq k$, we have

$$\sum_{s \in E} c_i |sE_i \Delta E_i| < \varepsilon c_i |E_i|.$$

Set $F = E_i$. Then for all $s \in E$, we have

$$\frac{|sF \Delta F|}{|F|} \leq \sum_{s \in E} \frac{|sF \Delta F|}{|F|} < \varepsilon$$

as desired.

For (3) $\Rightarrow$ (1), consider the set

$$I = \{(S, \varepsilon) \mid S \subset G \ \text{ finite and } \ \varepsilon > 0\}$$

is partially ordered by $(S_1,\varepsilon_1) \preceq (S_2,\varepsilon_2)$ if $S_1 \subset S_2$ and $\varepsilon_2 \leq \varepsilon_1$ (i.e., the amount of G covered is bigger and the tolerance is smaller).[40] For each $(S,\varepsilon) \in I$, let $F_{(S,\varepsilon)}$ be a Følner set for the pair, i.e.,

$$\max_{s\in S} \frac{|F_{(S,\varepsilon)}\Delta sF_{(S,\varepsilon)}|}{|F_{(S,\varepsilon)}|} < \varepsilon.$$

Set $m_{(S,\varepsilon)} := \frac{1}{|F_{(S,\varepsilon)}|}\chi_{F_{(S,\varepsilon)}} \in \ell^1(G)$ and $\hat{m}_{(S,\varepsilon)}$ as its alter-ego in $\ell^\infty(G)^*$.[41]

So, $\{\hat{m}_{(S,\varepsilon)}\}_{(S,\varepsilon)\in I}$ is a in net $\mathcal{S}(\ell^\infty(G))$. Recall from Exercise 7.6 (via the Banach-Alaoglu Theorem) that $\mathcal{S}(\ell^\infty(G))$ is compact in the weak*-topology, which means our net must have a weak*-convergent subnet $\{\hat{m}_j\}_{j\in J}$. We claim the weak*-limit m of this subnet is G-invariant.

First, we remark that for each fixed $s \in G$, the map $\phi \mapsto s\phi$ on $\ell^\infty(G)^*$ is weak*-continuous (since for each $g \in \ell^\infty(G)$, the map $\phi \mapsto s\phi(g) = \phi(s^{-1}g)$ is continuous by definition of the weak*-topology). It follows that for any $g \in \ell^\infty(G)$, we have

$$|m(g) - sm(g)| = \lim_j |\hat{m}_j(g) - s\hat{m}_j(g)|.$$

Now fix $s \in G$ and $g \in \ell^\infty(G)$, and write F_j and ε_j for the associated Følner set tolerance for each $\hat{m}_j$.

Then for each $j \in J$, we compute

$$\begin{aligned}
|\hat{m}_j(g) - s\hat{m}_j(g)| &= |\hat{m}_j(g) - \hat{m}_j(s^{-1}g)| = |\hat{m}_j(g - s^{-1}g)| \\
&= \left|\sum_{t\in G} m_j(t)\cdot(g(t) - s^{-1}g(t))\right| = \left|\sum_{t\in G} m_j(t)\cdot(g(t) - g(st))\right| \\
&= \frac{1}{|F_j|}\left|\sum_{t\in F_j} g(t) - g(st)\right| = \frac{1}{|F_j|}\left|\sum_{t\in F_j} g(t) - \sum_{t\in sF_j} g(t)\right|
\end{aligned}$$

The terms coming from $F_j \cap sF_j$ will cancel out, so we are actually just summing over the relative complements $F_j\backslash sF_j$ and $sF_j\backslash F_j$:

$$\begin{aligned}
\frac{1}{|F_j|}\left|\sum_{t\in F_j} g(t) - \sum_{t\in sF_j} g(t)\right| &= \frac{1}{|F_j|}\left|\sum_{t\in F_j\backslash sF_j} g(t) - \sum_{t\in sF_j\backslash F_j} g(t)\right| \leq \frac{1}{|F_j|}\left(\sum_{t\in F_j\backslash sF_j} |g(t)| + \sum_{t\in sF_j\backslash F_j} |g(t)|\right) \\
&= \frac{1}{|F_j|}\left(\sum_{t\in F_j\Delta sF_j} |g(t)|\right) \leq \frac{1}{|F_j|}|F_j\Delta sF_j|\|g\|_\infty \leq \varepsilon_j\|g\|_\infty
\end{aligned}$$

Since the nets were directed so that this subnet $(\varepsilon_j)_j$ converges to 0, this proves the claim. □

13.2. **Further Equivalent Formulations.** There are so many characterizations of amenability. Below we list a few that are used in operator algebras. One could write an entire book covering all the theory, ideas, and concepts that go into these. We will just direct you to where these are treated in the lecture.

Theorem 13.22. *Let G be a discrete group. Then the following are equivalent.*

(1) G is amenable, i.e., there exists a left-invariant mean $m \in \ell^\infty(G)^*$.
(2) There is a translation-invariant finitely additive probability measure on G.
(3) For any finite subset $E \subset G$ and $\varepsilon > 0$, there exists a finite mean f so that

$$\|f - s.f\|_1 < \varepsilon.$$

(4) G satisfies the Følner condition.
(5) $C^*_r(G)$ is nuclear.

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[40]We already say a similar set-up in the proof of Theorem 13.17. For any approximation statement in terms of finite subsets and small tolerances, this is a pretty common way to form a net capturing this information. If things are countable (or separable) one can usually get the job done with a nested sequence $F_1 \subset F_2 \subset F_3 \subset \ldots$ whose union is (dense in) the whole space and corresponding tolerances $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \ldots$ as we saw in the proof of Theorem 13.17.

[41]Here χ_A denotes the characteristic function on a set A.

(6) *The trivial representation (which sends all of G to $1 = 1_{B(\mathcal{H})}$) is weakly contained*[42] *in the regular representation λ. In other words, λ has almost invariant vectors. ****
(7) $\mathrm{C}^*_r(G) \cong \mathrm{C}^*(G)$.
(8) $\mathrm{C}^*_r(G)$ *has a character.*
(9)
(10) *(Fixed point property) any continuous affine action of G on a nonempty compact convex subset X of a locally convex topological vector space has a fixed point.*

Proof. For (1) $\iff$ (3) $\iff$ (4), see Theorem 13.21. For (1) $\iff$ (2), see Remark 13.3 and [7, Theorem 4.1.8 and Remark 4.3.5]. For (1) $\iff$ (5), see Theorem 13.17.

For (1) $\iff$ () FPP [2, Theorem G.1.7]. □

Remark 13.23. [2, Appendix G] addresses amenability for locally compact groups.

[42] See [2, Appendix F.1.1] and/or [2, Theorem F.4.4]

14. Hilbert C^*-modules and C^*-correspondences

Preview of Lecture: In lecture, we introduce Hilbert C^*-modules and some important examples, and lead to C^*-correpondences. In the next lecture, we introduce imprimitivity bimodules and the notion of Morita equivalence. We will conclude by sketching the proof that two C^*-algebras which have countable approximate units are Morita equivalent if and only if they are what's called *stably isomorphic.* This theorem is due to Brown, Green, and Rieffel (Theorem 1.2, [4]).

Throughout, A is a C^*-algebra and E is a right A-module with right action denoted $x \cdot a$ for $x \in E$ and $a \in A$.

Definition 14.1. An *A-valued semi-inner product* on E is a sesquilinear map $\langle \cdot, \cdot \rangle_A : E \times E \to A$ satisfying:

- $\langle x, y \cdot a + z \cdot b \rangle_A = \langle x, y \rangle_A a + \langle x, z \rangle_A b$ for all $x, y, z \in E$, $a, b \in A$
- $\langle x, y \rangle_A^* = \langle y, x \rangle_A$ for all $x, y \in E$
- $\langle x, x \rangle_A \geq 0$, as an element in A, for all $x \in E$.

We call E a *semi-inner product (right) A-module.*

Note that, by convention, we require A-linearity in the right component of the sesquilinear form. Thus, analogous to $\mathbb{C}$-valued inner products, we get that $\langle y \cdot b, x \rangle = b^* \langle y, x \rangle$ for all $x, y \in E$, $b \in A$. When it is clear that the sesquilinear form is A-valued, we drop the subscript from $\langle \cdot, \cdot \rangle_A$ and just write $\langle \cdot, \cdot \rangle$.

Fortunately, even in this far more general setting, we still have a Cauchy-Schwarz inequality.

Theorem 14.2 (Cauchy-Schwarz)**.** *If E is a semi-inner-product A-module, and $x, y \in E$, then*

$$\langle y, x \rangle \langle y, x \rangle^* \leq \| \langle x, x \rangle \| \langle y, y \rangle$$

Proof. If $x = 0$, there is nothing to show. Now, suppose $x \in E$ is such that $\| \langle x, x \rangle \| = 1$. For any $a \in A$,

$$\begin{array}{rcl} 0 & \leq & \langle y - x \cdot a, y - x \cdot a \rangle \\ & \leq & \langle y, y \rangle - \langle y, x \rangle a - a^* \langle x, y \rangle + a^* a \| \langle x, x \rangle \| \end{array} .$$

If we choose $a = \langle x, y \rangle = \langle y, x \rangle^*$, this yields the desired result. $\square$

Notice that this is a direct generalization of the Cauchy Schwarz inequality for Hilbert spaces.

Definition 14.3. An *A-valued inner product* on E is a sesquilinear map $\langle \cdot, \cdot \rangle_A : E \times E \to A$ satisfying the items in Definition 14.1 and positive-definiteness: $\langle x, x \rangle = 0$ if and only if $x = 0$. We call E an *inner product (right) A-module.*

Example 14.4. Every Hilbert space is a (left) inner-product $\mathbb{C}$-module.

Example 14.5. Every C^*-algebra A is an inner-product A module where the right action of A on itself is given by $a \cdot b = ab$ and $\langle a, b \rangle := a^* b$ for all $a, b \in A$.

Example 14.6. Let $J \triangleleft A$ be a closed two-sided ideal. Then J is an inner product A-submodule of A (as in Example 14.5).

For inner-product A-modules, we have the following useful inequality.

Proposition 14.7. *Let E be an inner-product A-module, and $x, y \in E$. Then*

$$\| \langle x, y \rangle \| \leq \| \langle x, x \rangle \|^{\frac{1}{2}} \| \langle y, y \rangle \|^{\frac{1}{2}} \tag{14.1}$$

Proof. To see this, notice that by the C^*-identity, the Cauchy-Schwarz inequality (for semi-inner product A-modules), and the fact that positive elements are norm preserving (i.e. $x \leq y$ implies $\|x\| \leq \|y\|$), we have

$$\| \langle x, y \rangle \|^2 = \| \langle x, y \rangle^* \langle x, y \rangle \| \leq \| \langle x, x \rangle \| \| \langle y, y \rangle \|,$$

as desired. $\square$

Proposition 14.8. *Let E be an inner product A-module, and for $x \in E$, define $\|x\|_E = \| \langle x, x \rangle_A \|^{\frac{1}{2}}$. Then $\| \cdot \|_E$ is a norm on E.*

Proof. Check that $\|\alpha x\|_E = |\alpha|\|x\|_E$ for all $\alpha \in \mathbb{C}$ and $x \in E$. For positive-definiteness, recall $\langle x, x\rangle \geq 0$, with equality if and only if $x = 0$; hence, $\|x\| \geq 0$, and $\|x\| = 0$ only when $\|\langle x, x\rangle\| = 0$, which by the norm on A implies that $\langle x, x\rangle = 0$, so that $x = 0$.

Last we check that $||\cdot||_E$ satisfies the triangle inequality. Take $x, y \in E$. Using Proposition 14.7 and the fact that the norm on A satisfies the triangle inequality, we have that

$$\|x+y\|_E^2 = \|\langle x,x\rangle_A + \langle x,y\rangle_A + \langle y,x\rangle_A + \langle y,y\rangle_A\| \leq \|x\|_E^2 + 2\|\langle x,y\rangle\| + \|y\|_E^2 \leq (\|x\|_E + \|y\|_E)^2$$

□

By definition of E, Equation 14.1 becomes $\|\langle x, y\rangle\| \leq \|x\|_E\|y\|_E$. If E is just a semi-inner product A-module, then the norm defined above will only be a semi-norm. In this case, we can construct a quotient of E that will be an inner product A-module. A direct consequence of Proposition 14.7 in the setting of a normed E is that $\langle\cdot,\cdot\rangle_A$ is continuous in both variables.

Exercise 14.9. Show that $N := \{x \in E : \langle x, x\rangle = 0\}$ is a norm-closed right A-submodule of E and that the sesquilinear form $[\cdot,\cdot] : E/N \times E/N \to A$ defined by $[x + N, y + N] := \langle x, y\rangle_A$ is positive-definite. Conclude that $\|x + N\| := \|\langle x, x\rangle_A\|^{1/2}$ defines an honest-to-goodness norm on E/N.

Exercise 14.10. For all $x \in E$,

$$\|x\|_E = \sup\{\|\langle x, y\rangle_A\| : y \in E, \|y\| \leq 1\}$$

Exercise 14.11. The norm on E also turns E into a normed A-module, i.e., for all $x \in E$, $a \in A$:

$$\|xa\| \leq \|x\|\|a\|.$$

If the norm $\|\cdot\|_E$ is complete, we say E is a *Hilbert A-module*. We say E is a *full* Hilbert A-module if the closure of the two-sided $*$-ideal $\langle E, E\rangle := \mathrm{Span}\{\langle x, y\rangle_A : x, y \in E\}$ of A is all of A.

The end of this subsection contains some technical results related to completing a right inner product module taking values in a pre-C^*-algebra to a Hilbert C^*-module. For the sake of smoothness to the introduction of Hilbert C^*-modules, you could skip to the next subsection and return here if/when you need to. We examine three scenarios where completion is needed. In each, A is a C^*-algebra:

- complete an inner product A-module E_0 to a Hilbert A-module
- complete a pre-C^*-algebra A_0 for which E_0 is an inner product A_0-module, and then complete E to be a Hilbert A-module.

Proposition 14.12. *Any inner product A-module E_0 can be completed to a Hilbert A-module.*

If A_0 is a pre-C^*-algebra (that is, not complete), then in a fashion very similar to that above, we can complete A_0 to a C^*-algebra A.

Proposition 14.13. *If A_0 is a pre-C^*-algebra and E is a Hilbert A_0-module, then the module structure on E_0 may be extended to A, the completion of A_0.*

Proposition 14.14. *Let A_0 be a pre-C^*-algebra, and E_0 be a pre-Hilbert module over A_0. Then we can complete the module action to a Hilbert A-module E, where A is the completion of A_0.*

14.1. **Important constructions of Hilbert C*-modules.** This subsection is devoted to exploring some important constructions involving Hilbert C^*-modules. Some examples are more abstract than others, but these abstract examples will be important for us in the final subsection of this section when we discuss Morita equivalence and stable isomorphism.

Definition 14.15. Let X and Y be right Hilbert A-modules. Their *direct sum* is the set $X \oplus Y := \{(x, y) : x \in X, y \in Y\}$ with right A-action $(x, y) \cdot a := (x \cdot a, y \cdot a)$ and A-valued inner product

$$\langle (x, y), (z, w)\rangle := \langle x, z\rangle + \langle y, w\rangle.$$

Exercise 14.16. Prove that the norm induced by the inner product defined on $X \oplus Y$ above is complete.

The next example will be important for the following chapter in discussing how Morita equivalence of two C^*-algebras is a way of saying that the two C^*-algebras have the same representation theory.

Example 14.17. Let $\pi : A \to B(\mathcal{H})$ be a representation and let E be a right Hilbert A-module with A-valued inner product $\langle \cdot, \cdot \rangle_A$. Define a $\mathbb{C}$-valued sesquilinear form on $E \odot \mathcal{H}$ by

$$(x \otimes h, y \otimes k) := \langle \pi(\langle y, x \rangle_A) h, k \rangle$$

for all $x \odot h, y \odot k \in E \odot \mathcal{H}$. Notice that the order in which x and y appear changes; this is because we usually want $\mathbb{C}$-valued inner products to be left-linear and right-conjugate linear, but our Hilbert A-module E has an A-valued inner product which is *right* A-linear; hence the switching of the positions of x and y. As an exercise, show that a nonzero element $\xi := (x \cdot a) \otimes h - x \otimes \pi(a)h$ satisfies $(\xi, \xi) = 0$, so $(\cdot, \cdot)$ is not positive definite.

In order to make $E \odot \mathcal{H}$ into a Hilbert space, we need to pass the sesquilinear form $(\cdot, \cdot)$ on $E \odot \mathcal{H}$ to a quotient of $E \odot \mathcal{H}$ in the usual way, and then we will complete this quotient inner product space in the norm induced by this inner product. The closed subspace of $E \odot \mathcal{H}$ that we need to quotient out by is precisely

$$\mathcal{N} := \{\xi \in E \odot \mathcal{H} : (\xi, \xi) = 0\} = \operatorname{Span}\{(x \cdot a) \otimes h - x \otimes \pi(a)h : a \in A, x \in E, h \in \mathcal{H}\}.$$

We denote the quotient of $E \odot \mathcal{H}$ by $\mathcal{N}$ as $E \odot_A \mathcal{H}$ to indicate that this tensor product is A-balanced (an element of A acting on the right of an element of E can be passed across the tensor product to act via π on the left of an element in $\mathcal{H}$). We denote the completion of $E \odot_A \mathcal{H}$ by $E \otimes_A \mathcal{H}$.

On the other hand, we can take the vector space tensor product of a right Hilbert A-module E with any Hilbert space $\mathcal{H}$ in the opposite order to produce a right Hilbert A-module: Define a right A-action on $\mathcal{H} \odot E$ by $(h \otimes x) \cdot a := h \otimes (x \cdot a)$ and an A-valued inner product $[\cdot, \cdot]_A$ on $\mathcal{H} \odot E$ by $[h \otimes x, k \otimes y]_A := \langle h, k \rangle \langle x, y \rangle_A$. Unlike the previous example, $[\cdot, \cdot]_A$ is already positive definite—no quotienting required.

Example 14.18. Let A be a C^*-algebra. The set

$$\mathsf{H}_A := \left\{ (a_i) \in \prod_{i=1}^{\infty} A : \sum_{i=1}^{\infty} a_i^* a_i \text{ converges in } A \right\}.$$

is a right Hilbert A-module with right A-action given by $(a_i) \cdot a := (a_i a)$ and $\langle (a_i), (b_i) \rangle_A = \sum_{i=1}^{\infty} a_i^* b_i$. We can think of this as $\ell^2(\mathbb{N}, A)$ which contains elements $a : \mathbb{N} \to A$, $i \mapsto a_i$, that are "square-summable" (here, the role of $|a_i|^2$ is being played by $a_i^* a_i$). This Hilbert A-module will play an important role in the next chapter. It has an interesting absorption property with other Hilbert A-modules: if E is any countably generated right Hilbert A-module, then $E \oplus \mathsf{H}_A$ and H_A are isomorphic as Hilbert A-modules (this is the **Kasparov Stabilization Theorem** [12, Theorem 5.49]).

Proposition 14.19. [12, 5.54] *If $\mathcal{H}$ is any separable Hilbert space and E is a countably generated and full right Hilbert A-module, then $\mathcal{H} \otimes E$ and H_A are isomorphic as right Hilbert A-modules.*

Recall that any separable Hilbert space $\mathcal{H}$ is isomorphic to $\ell^2(\mathbb{N})$, and if $\mathcal{H}'$ is any other Hilbert space, we know $\ell^2(\mathbb{N}) \otimes \mathcal{H}'$ is isomorphic to $\ell^2(\mathbb{N}, \mathcal{H}')$. Hence, $\mathcal{H} \otimes A$ can be naturally thought of as $\ell^2(\mathbb{N}, A)$, which we have just formally defined as H_A. But for **any** other full right Hilbert A-module E (which is countably generated!), why does $\mathcal{H} \otimes E$ have the same Hilbert A-module structure as $\mathcal{H} \otimes A$? The Kasparov Stabilization Theorem (KST) is needed to "shave off" any distinction between $\mathcal{H} \otimes A$ and $\mathcal{H} \otimes E$ as Hilbert A-modules. There is a Hilbert A-module Y such that $\mathcal{H} \otimes E \cong A \oplus Y$ (this is clever! see the construction below), and hence,

$$\mathcal{H} \otimes E \underset{\text{why?}}{\cong} \mathcal{H} \otimes (\mathcal{H} \otimes E) \cong \mathcal{H} \otimes (A \oplus Y) \cong (\mathcal{H} \otimes A) \oplus (\mathcal{H} \otimes Y) \cong \mathsf{H}_A \oplus (\mathcal{H} \otimes Y) \underset{\text{KST}}{\cong} \mathsf{H}_A.$$

How do we construct this magical Y? First, we use a lemma ([12, 5.53]) to choose a sequence $\{x_n\}_{n=1}^{\infty} \subset E$ such that $\sum_{n=1}^{\infty} \langle x_n, x_n \rangle_A$ converges to $\mathrm{id}_A \in M(A)$ (in the *strict topology* on $M(A)$ that we have not, and will not, introduce). Given an orthonormal basis $\{h_n : n \in \mathbb{N}\}$ for $\mathcal{H}$, define a map $T : A \to \mathcal{H} \otimes E$ by

$$T(a) := \sum_{n=1}^{\infty} h_n \otimes (x_n \cdot a),$$

which converges in $H \otimes E$ because

$$[T(a), T(a)]_A = \sum_{n=1}^{\infty} a^* \langle x_n, x_n \rangle_A a \to a^* \mathrm{id}_A a = a^* a.$$

One can check that T is adjointable and $T^*T(a) = a$ for all $a \in A$, and hence, T is a right Hilbert A-isomorphism of A onto its range. Though not all closed submodules of Hilbert modules are complementable, the closed submodule $\operatorname{ran}(T)$ of $\mathcal{H}\otimes E$ is complementable, and its orthogonal complement is $\ker(T^*)$. Choose $Y := \ker(T^*)$, so that, as right Hilbert A-modules, we have the following isomorphisms:

$$\mathcal{H} \otimes E \cong \operatorname{ran}(T) \oplus \ker(T^*) \cong A \oplus Y.$$

Example 14.20. The motivating example for the study of Hilbert C^*-modules in their own right comes from differential geometry, and thus arises in the closely-related field of noncommutative geometry. Let $A = C(X)$, where X is a compact, Hausdorff space. A vector bundle E can be defined as follows. Take a fixed euclidean space H. For each $t \in X$, set H_t to be a subspace of H. Let $E \subset C(X, H)$ such that for every $t \in X$, and $\eta \in E$, $\eta(t) \in H_t$. We may endow E with an 'A-valued' inner product given by

$$\langle \eta, \xi\rangle(t) = \langle \eta(t), \xi(t)\rangle_H$$

That is, $\langle \eta, \xi\rangle \in A$. We show this is an inner product. By virtue of the inner product on H, we have that for each $t \in X$ that

$$\langle \eta, \xi\rangle(t) = \langle \eta(t), \xi(t)\rangle_H = \overline{\langle \xi(t), \eta(t)\rangle_H} = \overline{\langle \xi, \eta\rangle(t)}$$

Thus, $\langle \eta, \xi\rangle = \overline{\langle \xi, \eta\rangle}$. The other properties may checked in a similar fashion. Additionally, we have that E is an A-module with action defined point-wise. That is, if $\eta \in E$ and $f \in C(X)$, then $(\eta f)(t) := \eta(t)f(t)$. Since $\eta(t) \in H_t$, and $f(t) \in \mathbb{C}$, we have that $\eta(t)f(t) \in H_t$ for each t so that ηf is indeed an element of E.

On the other hand, the motivating example for using Hilbert C^*-modules (actually, *C^*-correpsondences* to define Morita equivalence of C^*-algebras, is the following.

Example 14.21 ([12], p. 14)**.** Let G be a discrete group. The following example still works when G is locally compact, but for the sake of simplicity, we will stick to the discrete setting. Let H be a closed subgroup of G. We will see that $C^*(G)$ is a right Hilbert $C^*(H)$-module by first defining a right action by $C_c(H)$ and $C_c(H)$-valued inner product on $C_c(G)$. We will then need to use a completion argument to obtain the desired $C*(H)$-Hilbert module structure on $C^*(G)$.

Given $f, g \in C_c(G)$ and $b \in C_c(H)$, define for $s \in G$

$$(f \cdot b)(s) := \sum_{t\in H} f(st^{-1})b(t)$$

and for $r \in H$

$$\langle f, g\rangle_{C_c(H)}(r) := \sum_{t\in H} \overline{f(r)}g(rs).$$

It is an exercise to show that for all $f, g \in C_c(H)$ and $b \in C_c(G)$,

- $f \cdot b \in C_c(G)$
- $\langle f, g\rangle_{C_c(H)} \in C_c(G)$ for all $f, g \in C_c(G)$, $b \in C_c(H)$
- $\langle \cdot, \cdot\rangle_{C_c(H)}$ is $C_c(H)$-sesquilinear
- $\langle f, f\rangle_{C_c(H)}$ is a positive element *in the C^*-completion of $C_c(H)$ to $C^*(H)$.*

Then the completion of $C_c(G)$ to $C^*(G)$ is a right Hilbert $C^*(H)$-module.

14.2. **Adjointable Maps.** Throughout, both E and F are (right) Hilbert A-modules. We will not distinguish between their A-valued inner products notationally, but do keep in mind that they may differ.

Definition 14.22. A (not necessarily linear, not necessarily bounded) map $T : E \to F$ is *adjointable* if there exists a (not necessarily linear, not necessarily bounded) map $S : F \to E$ such that for all $x, y \in E$,

$$\langle Tx, y\rangle = \langle x, Sy\rangle.$$

Denote the collection of all adjointable maps from E to F by $\mathcal{L}(E, F)$. When $E = F$, we denote $\mathcal{L}(E, F)$ by $\mathcal{L}(E)$.

Proposition 14.23. *If $T \in \mathcal{L}(E, F)$, then T is A linear and bounded.*

Proof. Let $x, y \in E$ and $a \in A$, then

$$\langle x, T^*(y \cdot a)\rangle = \langle Tx, y \cdot a\rangle = \langle Tx, y\rangle a = \langle x, T^*(y) \cdot a\rangle$$

Hence, $T^*(y \cdot a) = T^*(y) \cdot a$. For boundedness, fix $x \in E$ and consider $f_x : F \to A$ defined by

$$f_x(y) = \langle Tx, y\rangle$$

for all $y \in F$. If $||x||_E \leq 1$, then

$$\|f_x(y)\| = \|\langle Tx, y\rangle\| = \|\langle x, T^*y\rangle\| \leq ||x|| \|T^*y\| \leq ||T^*y||.$$

Hence, for fixed $y \in E$, we have

$$\sup_{\{x \in E: ||x||_E \leq 1\}} \|f_x(y)\| \leq \|T^*y\|.$$

By the uniform boundedness principal,

$$\sup_{\{x \in E: ||x||_E \leq 1\}} \|f_x\| = \sup_{\{x \in E: ||x||_E \leq 1\}} \sup_{\|y\|=1} \|f_x(y)\| < \infty$$

So, we have that there is some M such that

$$\left\| f_x\left(\frac{Tx}{\|Tx\|}\right)\right\| \leq M$$

In other words, $\|Tx\| \leq M$. Taking sup over all $\|x\| \leq 1$ yields result. □

When $A = \mathbb{C}$, every right Hilbert A-module is a Hilbert space, and $\mathcal{L}(\mathcal{H})$ is precisely $B(\mathcal{H})$. However, this is not true when A is something more general than $\mathbb{C}$; not every bounded A-linear map is adjointable (in general, $\mathcal{L}(E) \subsetneq B(E)$.)

Proposition 14.24. *$\mathcal{L}(E)$ is a C^*-algebra.*

Proof. First, note that if $T, S \in \mathcal{L}(E)$, then $TS \in \mathcal{L}(E)$, as

$$\langle TSx, y\rangle = \langle x, S^*T^*y\rangle$$

Hence, $(TS)^* = S^*T^*$. The other axioms for a $*-$algebra hold in a similar fashion. Moreover, $\mathcal{L}(E)$ satisfies the C^* identity. To see this, note that if $T \in B(E)$, that

$$\|\langle Tx, x\rangle\|^2 \leq \|\langle Tx, Tx\rangle\| \|\langle x, x\rangle\| \leq \|T\|^2 \|x\|^2$$

for all $x \in E$. Hence, we have by the C^*-identity that

$$\|T\| \geq \sup_{E_1} \|\langle Tx, x\rangle\|$$

where E_1 is the unit ball in E. Applying this result to T^*T for $T \in \mathcal{L}(E)$, we find that

$$\|T^*T\| \geq \sup_{E_1} \|\langle T^*Tx, x\rangle\| = \|T\|^2$$

Hence, $\|T^*T\| = \|T\|^2$ for all $T \in L(E)$. In particular,

$$\|T^*\|^2 = \|TT^*\| \leq \|T\| \|T^*\| \quad \text{and} \quad \|T\|^2 = \|T^*T\| \leq \|T^*\| \|T\|$$

So that $\|T\| = \|T^*\|$. Finally, to show that $\mathcal{L}(E)$ is complete, we show it is norm closed in $B(H)$. Suppose that T is a limit point of $\mathcal{L}(E)$. Say, $T_n \to T$ with $T_n \in \mathcal{L}(E)$. Notice that

$$\|T_n^* - T_m^*\| = \|T_n - T_m\| \to 0$$

so that T_n^* forms a Cauchy sequence. Hence, it converges to some element $S \in B(E)$. By continuity of $\langle \cdot, \cdot\rangle$ in both variables, we get

$$\langle Tx, y\rangle = \lim_n \langle T_n x, y\rangle = \lim \langle x, T_n^* y\rangle = \langle x, Sy\rangle$$

so that $S = T^*$. So in fact, $T \in \mathcal{L}(E)$. □

Last, but certainly not least, is that the structure of a C^*-algebra A as a right Hilbert A-module gives rise to its multiplier algebra, which is the "largest" unital C^*-algebra in which we can embed A while still having A as an essential ideal within that larger C^*-algebra. We will say a bit more about the multiplier algebra for a C^*-algebra later in this section.

Definition 14.25. Given $y \in E$ and $x \in F$, define $\theta_{y,x} : E \to F$ by $\theta_{y,x}(z) := y \cdot \langle x, z\rangle_A$ for all $z \in E$.

Exercise 14.26. Compute the adjoint and operator norm of $\theta_{y,x}$.

Exercise 14.27. For $x, y, z, w \in E$, simplify the composition of the operators $\theta_{y,x} \circ \theta_{w,z}$ to $\theta_{\xi,\eta}$ for some $\xi, \eta \in E$.

When $\mathcal{H}$ and $\mathcal{H}'$ are Hilbert spaces, a linear operator $T : \mathcal{H} \to \mathcal{H}'$ is *compact* if the closure of the image of the unit ball in $\mathcal{H}$ under T is compact in $\mathcal{H}'$. It turns out that the collection of all compact operators $K(\mathcal{H}, \mathcal{H}')$ forms a closed two-sided ideal and the collection of rank-one operators of the form $h \otimes k^*$ span a dense subspace of $K(\mathcal{H}, \mathcal{H}')$. When investigating an analogue for compact operators between Hilbert C^*-modules, we will generalize the latter notion of compactness.

Definition 14.28. Let E and F be Hilbert A-modules. The collection of *compact operators from E to F* is defined to be $K(E, F) := \overline{\text{Span}\{\theta_{y,x} : x \in E, y \in F\}}$.

Exercise 14.29. For any Hilbert A-module E, $K(E)$ is a closed two-sided ideal in $\mathcal{L}(E)$.

Example 14.30. Let A be considered as a right Hilbert A-module. Then $K(A)$ and A are isomorphic as C^*-algebras. Indeed, let $\phi : K(A) \to A$ be given by $\phi(\theta_{a,b}) = ab^*$. By a previous exercise,

$$\|\phi(\theta_{a,b})\| = \|ab^*\| = \|\theta_{a,b}\|,$$

so ϕ extends continuously to an isometry on $K(A)$. over the set $\{\theta_{a,b}\}$. Since products are dense in A (by virtue of A having an approximate identity), we may conclude that the image of ϕ is dense in A. Hence, ϕ is onto. It remains to prove that ϕ is a $*$-homomorphism, which is left as an exercise.

This example highlights another important reason we care about Hilbert C^*-modules and their adjointable maps: any C^*-algebra A can be realized as the ideal $K(A)$ inside the **unital** C^*-algebra $\mathcal{L}(A)$. For this reason, $\mathcal{L}(A)$ is referred to as the *multiplier algebra of* A and is often denoted $M(A)$. Murphy introduces the multiplier algebra for a C^*-algebra differently–without using the theory of Hilbert C^*-modules. Indeed, $M(A)$ is also the *double centralizer* of A. We will not say more about this construction here, but you should know there is another realization of $M(A)$ which is independent from Hilbert C^*-module theory.

We have already seen a unitization $\widetilde{A}$ of a C^*-algebra A in Definition 1.16. Recall $\widetilde{A} = A \oplus \mathbb{C}$ with a multiplication that makes the copy of $1 \in \mathbb{C}$ inside of $\widetilde{A}$ a unit for $\widetilde{A}$. This construction is the smallest unitization for A in the sense that its quotient by A is isomorphic to $\mathbb{C}$. However, the multiplication in $\widetilde{A}$ causes A to not be an ideal in $\widetilde{A}$. So, what makes $M(A)$ an ideal unitization (pun intended here) is that $M(A)$ does contain A as an ideal, $A \cong K(A)$, and moreover, $M(A)$ the "largest" unitization of A that still contains A as an *essential ideal*.

Definition 14.31. A closed ideal I of a C^*-algebra A is *essential* if for all $a \in A$, $aI = \{0\}$ implies $a = 0$ (equivalently, $Ia = \{0\}$ implies $a = 0$).

Exercise 14.32. Let A be a C^*-algebra. Show that $K(A)$ is an essential ideal of $\mathcal{L}(A)$.

Theorem 14.33 ([11], 3.1.8)**.** *Given a closed ideal I in a C^*-algebra A, there is a unique $*$-homomorphism $\varphi : A \to M(I)$ extending the inclusion $I \hookrightarrow M(I)$. Moreover, φ is injective if I is an essential ideal in A.*

Proof. Let I be a closed ideal of A and let $\iota : I \to M(I)$ be the inclusion given by the isomorphism $I \cong K(I)$ in Example 14.30. Define $\varphi : A \to M(I)$ on $a \in A$ by $\varphi(a)x = ax$ for all $x \in I$. Then $\varphi|_I = \iota$ and φ is a $*$-homomorphism.

Now, suppose $\psi : A \to M(I)$ is another $*$-homomorphism which satisfies $\psi(x) = \iota(x)$ for all $x \in I$. Let $a \in A$. Then $\varphi(ax) - \psi(ax) = \iota(ax) - \iota(ax) = 0$ for all $x \in I$, and thus, $\varphi(a)x - \psi(a)x = 0$ for all $x \in I$, so $\varphi(a) = \psi(a)$. Therefore, $\varphi = \psi$. If φ is injective, then for each nonzero element $a \in A$ there exists $x \in I$ such that $\varphi(a)x \neq 0$. This is precisely the contrapositive of the definition for I to be essential in A. ☐

Exercise 14.34. From the above theorem, discuss why it is preferable for a unitization of a C^*-algebra A to contain A as an ideal. Why is it nicer for that unitization to contain A as an essential ideal?

14.3. **C*-correspondences.** Throughout, A and B are a C*-algebras and E is a (right) Hilbert A-module.

Definition 14.35. A right Hilbert A-module E is a *B-A-correspondence* if there is a $*$-representation $\phi : B \to \mathcal{L}(E)$, i.e., E comes equipped with a left action of B via adjointable (and thus right A-linear) operators. Specifically, for all $x, y \in E$, $b \in B$, and $a \in A$,

$$\langle \phi(b)x, y\rangle_A = \langle x, \phi(b^*)y\rangle_A,$$

which implies

$$\phi(b)(x \cdot a) = (\phi(b)x) \cdot a.$$

A B-A-correspondence E is *nondegenerate* if $\phi(B)E$ is dense in E.

Example 14.36. Let $\mathcal{H}$ be a Hilbert space with a right-linear inner product. Then $\mathcal{H}$ is a $K(\mathcal{H})$-$\mathbb{C}$-correspondence where:

- $\forall h \in \mathcal{H}$, $\lambda \in \mathbb{C}$, define $h \cdot \lambda$ to be λh,
- $\phi : K(\mathcal{H}) \to B(\mathcal{H})$ is the inclusion map.

When E is a B-A-correspondence, you may see it denoted in the literature as ${}_BE_A$. We will adopt that notation in these notes when it is convenient to do so. Any B-A-correspondence ${}_BE_A$ provides a linking structure to induce a representation of B from a representation of A. This is where the construction in Example 14.17 comes into play:

Proposition 14.37. *Let ${}_BE_A$ be a B-A-correspondence and $\pi : A \to B(\mathcal{H})$ a $*$-representation.*

- *If π is nondegenerate and E is nondegenerate as a Hilbert B-module ($B \cdot E$ is dense in E), then the induced representation $\widetilde{\pi} : B \to B(E \otimes_A \mathcal{H})$ given by $\widetilde{\pi}(b)(x \otimes h) := \phi(b)x \otimes h$ is nondegenerate.*
- *If π is faithful and B acts faithfully on E, then the induced representation $\widetilde{\pi}$ of B is faithful.*

Remark 14.38. If A is just $\mathbb{C}$, then E_A is an honest-to-goodness Hilbert space and the inner product defined above on $E_A \odot \mathcal{H}$ is the inner product defined on the tensor products of vector spaces in the Prereq notes (indeed, what options are there for nondegenerate representations of $\pi : A \to B(\mathcal{H})$ when $A = \mathbb{C}$?).

Proposition 14.39. *Consider A as a right Hilbert A-module in the usual way. Then A is an $\mathcal{L}(A)$-A-correspondence. Consequently, any nondegenerate representation of A induces a representation of its multiplier algebra $M(A)$.*

14.4. **Imprimitivity bimodules and induced representations.** Starting from a right Hilbert A-module E, we added a left action of another C^*-algebra B via adjointable maps on $L(E)$ to obtain a C^*-correspondence. Now we will study C^*-correspondences which have even *more* structure: imprimitivity bimodules. Our first example revisits Example 14.36.

Definition 14.40. Let A and B be C^*-algebras. A *B-A-imprimitivity bimodule* is a B-A-bimodule E such that

(1) E is a full left Hilbert B-module and a full right Hilbert A-module
(2) for all $x, y \in E$, $a \in A$, $b \in B$,

$$\langle b \cdot x, y \rangle_A = \langle x, b^* \cdot y \rangle_A \quad \text{and} \quad {}_B\langle x \cdot a, y \rangle = {}_B\langle x, y \cdot a^* \rangle$$

(3) for all $x, y, z \in E$,

$$_B\langle x, y \rangle \cdot z = x \cdot \langle y, z \rangle_A$$

Condition (2) says that B acts as adjointable operators on E as a right Hilbert A-module and vice-versa. Essentially, a B-A-imprimitivity bimodule is both a B-A-correspondence and a flip-flopped A-B-correspondence, so that we are not only able to take representations of A and induce representations of B, but we are also able to take representation of B and induce representations of A.

Exercise 14.41. Prove that A is a **full** right Hilbert A-module.

Example 14.42. We show that every Hilbert space $\mathcal{H}$ is a $K(\mathcal{H})$-$\mathbb{C}$-imprimitivity bimodule. Recall the set up from Exercise 14.36. Define a $K(\mathcal{H})$-valued sesquilinear form on $\mathcal{H}$ by

$$_{K(\mathcal{H})}\langle h, k \rangle := h \otimes k^* \quad \text{where} \quad h \otimes k^*(f) := \langle f, k \rangle_{\mathbb{C}}\, h.$$

It's nontrivial to check the details that $\mathcal{H}$ is a full left Hilbert $K(\mathcal{H})$-module. See Raeburn-Williams for the details. As an exercise, check that $\mathcal{H}$ satisfies the Condition (3) from Definition 14.40 necessary to be a $K(\mathcal{H})$-$\mathbb{C}$-correspondence.

To emphasize, it is not particularly interesting to recast a Hilbert space $\mathcal{H}$ as a $K(\mathcal{H})$-$\mathbb{C}$-imprimitivity module; rather, it is extremely interesting to understand that in this role, the Hilbert space $\mathcal{H}$ acts as a conduit to pass representations between $K(\mathcal{H})$ and $\mathbb{C}$.

Proposition 14.43. *Let $\mathcal{H}, \mathcal{H}'$ be Hilbert spaces. Every nondegenerate representation $\pi : K(\mathcal{H}) \to B(\mathcal{H}')$ is a direct sum of the identity representation, i.e., $\mathcal{H}'$ is isomorphic to the direct sum of some number of copies of $\mathcal{H}$.*

How can we use C^*-correspondences to better understand representation theory for more general C^*-algebras than just $K(\mathcal{H})$? It turns out that a (large) generalization of Example 14.42 is true.

Example 14.44. Every full right Hilbert A-module E is a $K(E)$-A-imprimitivity bimodule. Here, the inner product ${}_{K(E)}\langle \cdot, \cdot \rangle$ is defined on $x, y \in E$ by ${}_{K(E)}\langle x, y \rangle := \theta_{x,y}$.

Consequently, the collection of nondegenerate representations of A are directly tied to the collection of full right Hilbert A-modules.

Exercise 14.45. Given a right Hilbert A-module E, prove that the inner products ${}_{K(E)}\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_A$ satisfy Condition (3) of Definition 14.40.

Proposition 14.46. *Given an B-A-imprimitivity bimodule, the left action of B as adjointable operators on E given by $\phi : B \to \mathcal{L}(E)$ is an isomorphism of B onto $K(E)$ which satisfies*

$$\phi({}_B\langle x, y \rangle) = {}_{K(E)} \langle x, y \rangle.$$

Proof. By hypothesis, ϕ is a $*$-homomorphism. Fix $x, y \in E$. For any $z \in E$,

$$\phi({}_B\langle x, y \rangle)z = {}_B \langle x, y \rangle \cdot z = x \cdot \langle y, z \rangle_A = \theta_{x,y}(z) = {}_{K(E)} \langle x, y \rangle z.$$

Therefore, $\phi({}_B\langle x, y \rangle) = {}_{K(E)} \langle x, y \rangle$. Now, we have that $\phi({}_B\langle E, E \rangle) = \text{Span}\{\theta_{x,y} : x, y \in E\}$. As E is a full left Hilbert B-module, the norm closure of ${}_B\langle E, E \rangle$ is all of B. Moreover, ϕ is norm-continuous, so

$$\phi(B) = \phi(\overline{{}_B\langle E, E \rangle}) = \overline{\phi({}_B\langle E, E \rangle)} = \overline{\text{Span}\{\theta_{x,y} : x, y \in E\}} = K(E).$$

We leave it as an exercise to show ϕ is injective. □

14.5. **Morita Equivalence and Complementary Full Corners.**

Remark 14.47. You are aught to hear folks in the operator algebras community say that two C^*-algebras which are Morita equivalent "have the same representation theory." The following definition coupled with the previous subsection makes this notion clear.

Definition 14.48. Two C^*-algebras A and B are *Morita equivalent* if there exists a B-A-imprimitivity bimodule.

Theorem 14.49. *Morita equivalence is an equivalence relation on C^*-algebras.*

We will not show the details of the proof of Theorem 14.49, though reflexivity is proven in the following exercise. The details of the proof, particularly transitivity of Morita equivalence, can be found in Raeburn-Williams, Proposition 3.18.

Exercise 14.50. Prove that every C^*-algebra A is Morita equivalent to itself.

Exercise 14.51. If E is a B-A-imprimitivity bimodule and F is a C-B-imprimitivity bimodule, how would you construct a C-A imprimitivity bimodule? If you have a candidate in mind, check the details. Conclude that Morita equivalence is a transitive relation.

In [12], the authors remark that the following theorem's proof is as important as its result. Indeed, their proof requires an equivalent definition of Morita equivalence; it utilizes a 2×2-matrix "trick" to change a statement about C^*-correspondences into a statement about C^*-algebras.

Definition 14.52. Let E be a B-A-imprimitivity bimodule, and let $\widetilde{E}$ denote the *dual of* E, which is E with the sides of the actions of A and B switched, so that $\widetilde{E}$ is an A-B-imprimitivity bimodule. Set $\mathsf{M} := E \oplus A$, and for each $a \in A, b \in B$ and $x, y \in E$ let

$$L = \begin{pmatrix} b & x \\ \widetilde{y} & a \end{pmatrix}$$

denote the map from M to M which acts on an element $(z, c) \in M$ by

$$L(z, c) = \begin{pmatrix} b & x \\ \widetilde{y} & a \end{pmatrix} \begin{pmatrix} z \\ c \end{pmatrix} = \begin{pmatrix} b \cdot z + x \cdot c \\ \langle y, z \rangle_A + ac \end{pmatrix}$$

The following exercise asks you to verify that L is adjointable and to calculate its adjoint. The collection of all operators $\mathcal{C}$ on M form a $*$-subalgebra of $\mathcal{L}(\mathsf{M})$, which we call the *linking algebra* of the imprimitivity bimodule ${}_BE_A$.

Exercise 14.53. Given $a \in A, b \in B, x, y \in E$, prove that $L : \mathsf{M} \to \mathsf{M}$ defined above is adjointable, and compute its adjoint.

Exercise 14.54. Prove that the collection of all elements $L \in \mathcal{L}(\mathsf{M})$ form a $*$-subalgebra.

Lemma 14.55 (3.20,[12])**.** *If $L \in \mathcal{L}(\mathsf{M})$ is given as above, then*

$$\max\{\|a\|, \|x\|_A, \|y\|_B, \|b\|\} \leq \|L\| \leq \|a\| + \|x\|_A + \|y\|_B + \|b\|$$

This lemma establishes that $\mathcal{C}$ is, in fact, a C^*-subalgebra of $\mathcal{L}(\mathsf{M})$. What's so important about this C^*-algebra $\mathbb{C}$? Notice that its top left corner contains an isomorphic copy of B and its bottom right corner contains an isomorphic copy of A.

Definition 14.56. Let $\mathcal{C}$ be a C^*-algebra.

- C^*-subalgebra A of $\mathcal{C}$ is a *corner* if there is a projection $p \in M(\mathcal{C})$ such that $A = p\mathcal{C}p$.
- A corner A of $\mathcal{C}$ is *full* if it is not contained in any proper closed two-sided ideal of $\mathcal{C}$. Equivalently, if $A = p\mathcal{C}p$ for some projection $p \in \mathcal{C}$, then A is *full* if and only if $\operatorname{Span}\{\mathcal{C}p\mathcal{C}\}$ is dense in $\mathcal{C}$.
- If A and B are corners of $\mathcal{C}$ such that $A = p\mathcal{C}p$ and $B = q\mathcal{C}q$ for some projections $p, q \in M(\mathcal{C})$, then A and B are *complementary corners* if $p + q = 1 \in M(\mathcal{C})$.

Exercise 14.57. Let $\mathcal{C}$ be a C^*-algebra and A a C^*-subalgebra of $\mathcal{C}$. Prove that A is a full corner of $\mathcal{C}$ if and only if there is a projection $p \in M(\mathcal{C})$ such that $A = p\mathcal{C}p$ and $\operatorname{Span}\{\mathcal{C}p\mathcal{C}\}$ is dense in $\mathcal{C}$.

A simple example (no pun intended) of a C^*-algebra containing two complementary full corners is when $\mathcal{C} = M_2(\mathbb{C})$.

Example 14.58. Consider

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathbb{C} \right\} \quad \text{and } B = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} : b \in \mathbb{C} \right\}.$$

Consider the projections E_{11} and E_{22} in $M_2(\mathbb{C})$, which is its own multiplier algebra as $M_2(\mathbb{C})$ is unital. Moreover, $A = E_{11}M_2(\mathbb{C})E_{11}$, $B = E_{22}M_2(\mathbb{C})E_{22}$, so A and B are corners. Since $M_2(\mathbb{C})$ is simple, A and B are automatically full corners, and, finally, $E_{11} + E_{22} = I_2$, so A and B are complementary.

If we consider $\mathbb{C}$ as a $\mathbb{C}$-$\mathbb{C}$-imprimitivity bimodule, we can actually view the C^*-subalgebra of $M_2(\mathbb{C})$ below as its linking algebra:

$$\mathcal{C} = \left\{ \begin{pmatrix} b & x \\ \overline{y} & a \end{pmatrix} : x, y, a, b \in \mathbb{C} \right\}$$

Here, then, we are viewing A and B as isomorphic copies of $\mathbb{C}$ inside of $M_2(\mathbb{C})$.

Theorem 14.59 (Brown-Green-Rieffel, 1.1 [4])**.** *Let A and B be C^*-algebras. Then A and B are Morita equivalent if and only if there is a C^*-algebra $\mathcal{C}$ with complementary full corners isomorphic to A and B, respectively.*

Sketch. Suppose A and B are complementary full corners of a C^*-algebra $\mathcal{C}$. Then there exist projections $p, q \in M(\mathcal{C})$ satisfying $p + q = 1$ such that $A = p\mathcal{C}p$ and $B = q\mathcal{C}q$ are not contained in any nontrivial two-sided ideal of $\mathcal{C}$. It is an exercise to show that $q\mathcal{C}p$ is a B-A-imprimitivity bimodule, and hence, A and B are Morita equivalent.

Conversely, suppose A and B are Morita equivalent, so there exists a B-A-imprimivity bimodule ${}_BE_A$. Let $\mathcal{C}$ be the linking algebra for ${}_BE_A$. We want to choose p and q in an analogous way to E_{11} and E_{22} as in Example 14.58. Denote by $\mathrm{id}_A \in \mathcal{L}(A)$ and $\mathrm{id}_B \in L(B)$ the identity maps on A and B, respectively, and set

$$q = \begin{pmatrix} \mathrm{id}_B & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad p = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{id}_A \end{pmatrix}$$

so that $q\mathcal{C}q = B$ and $p\mathcal{C}p = A$. Note that these satisfy $p + q = \mathrm{id}_{\mathsf{M}}$, the identity in $\mathcal{L}(\mathsf{M})$. We do need to ensure that p and q actually reside within the multiplier algebra $M(\mathcal{C})$, and, *a priori*, we only know that they are in $\mathcal{L}(\mathsf{M})$. A short but technical argument ensures that p and q are, in fact, in $M(\mathcal{C})$, but we will not give the proof here; see [12, Proposition 2.53] for the details.

Thus far we have established that (isomorphic copies of) A and B are complementary corners of the linking algebra $\mathcal{C}$ for ${}_BE_A$, so it remains to show that A and B are full. By Exercise 14.57, it suffices to show that the spans of $\mathcal{C}q\mathcal{C}$ and $\mathcal{C}p\mathcal{C}$ are dense in $\mathcal{C}$. The details of this approximation argument can be found in the proof of [12, Theorem 3.19]. □

While the above theorem is due to Brown-Green-Rieffel, the proof outlined here is taken from [12, Theorem 3.19] due to its streamlined argument of why the linking algebra for an imprimitivity bimodule is, in fact, a C^*-algebra.

Exercise 14.60. Work through the details of showing A in the above theorem is a full corner of $\mathcal{C}$.

14.6. **The Brown-Green-Rieffel Theorem.** What is the purpose of reframing Morita equivalence of two C^*-algebras in this way? Though Theorem 14.59 is interesting and useful in its own right, the main theorem of this section is the following, and its proof requires this reformulation of Morita equivalence:

Theorem 14.61 (Brown-Green-Rieffel, 1.2 [4])**.** *Let A and B be C^*-algebras. If A and B are stably isomorphic, then they are Morita equivalent. Conversely, if A and B are Morita equivalent and both possess strictly positive elements, then they are stably isomorphic.*

It turns out that the presence of a strictly positive element in a C^*-algebra implies that the C^*-algebra has a countable approximate identity. Raeburn and Williams (and others) refer to the property of a C^*-algebra having a countable approximate identity as *σ-unital.* For the sake of being explicit, we will continue to say "has a countable approximate identity."

Definition 14.62. Two C^*-algebras A and B are *stably isomorphic* if there is an isomorphism between $A \otimes \mathcal{K}$ and $B \otimes \mathcal{K}$.

Remark 14.63. The C^*-algebra of compact operators $\mathcal{K}$ on a separable Hilbert space is nuclear. Consequently, these tensor product C^*-algebras have unique tensor C^*-norms by Theorem 12.3.

Recall that the representation theory of $\mathcal{K}$ is effectively trivial (Proposition 14.43), so given a C^*-algebra A, how do we expect the representations of $A \otimes \mathcal{K}$ to compare to those of A? They should be... "the same!"

Exercise 14.64. This exercise helps you to prove that A is Morita equivalent to $A \otimes \mathcal{K}$, and relies heavily on Proposition 14.14. Let $\mathcal{H}$ be a separable Hilbert space. Recall that A and $\mathcal{H}$ are A-A- and $K(\mathcal{H})$-$\mathbb{C}$-imprimitivity bimodules, respectively. For $a \otimes h \in A \odot \mathcal{H}$, define a left $A \odot \mathcal{K}$-action by $(b \otimes T)(a \otimes h) := ba \otimes Th$, and define a right action of $A \odot \mathbb{C}$ on $A \odot \mathcal{H}$ by $(a \otimes h) \cdot (c \otimes \lambda) := ac \otimes \lambda h$. For $a \otimes h, b \otimes k \in A \odot \mathcal{H}$, define an $A \otimes K(\mathcal{H})$-valued inner product by

$$ {}_{A\otimes\mathcal{K}}\langle a \otimes h, b \otimes k\rangle :={}_A \langle a, b\rangle \otimes_{K(\mathcal{H})} \langle h, k\rangle $$

and an $A \otimes \mathbb{C}$-valued inner product by

$$ \langle a \otimes h, b \otimes k\rangle_{A\otimes\mathbb{C}} := \langle a, b\rangle_A \otimes \langle h, k\rangle_{\mathbb{C}}. $$

(1) Determine what details need to be checked in order to ensure that these left and right actions by $A \odot \mathcal{K}$ and $A \odot \mathbb{C}$ extend to actions by the C^*-completions $A \otimes \mathcal{K}$ and $A \otimes \mathbb{C}$, respectively. If you're feeling spicy, check these details, in detail.
(2) Determine what details need to be checked in order to ensure that the norms induced by the above $A \otimes \mathcal{K}$- and $A \otimes \mathbb{C}$-valued inner products on $A \odot \mathcal{H}$ give rise to the completion $A \odot \mathcal{H}$ in this induced norm being a full left $A \otimes \mathcal{K}$-module and a full right $A \otimes \mathbb{C}$-module, respectively.
(3) Check conditions (2) and (3) of Definition 14.40 to deduce that $A \otimes \mathcal{H}$ is a $A \otimes \mathcal{K}$-$A \otimes \mathbb{C}$-imprimitivity bimodule.
(4) Conclude that $A \otimes \mathcal{K}$ and A are Morita equivalent.

Exercise 14.65. Let X and Y be Hilbert A- and B-modules, respectively. Then $X \otimes Y$ is a Hilbert $A \otimes_{\min} B$-module and

$$ K(X \otimes Y) \cong K(X) \otimes_{\min} K(Y). $$

Instead of proving these statements in great detail, discuss why the isomorphism above involves the minimal (spatial) tensor product norm. This begs a second discussion question: in general, is $K(E)$ nuclear for any Hilbert A-module E?

It is now time to prove the main goal of this section. The forward direction is not surprising; stable isomorphism is a stronger equivalence relation than Morita equivalence. The reverse implication requires rather a lot of technical lemmas and propositions to prove, and it is both surprising and useful. In their original 1977 paper, Brown, Green, and Rieffel assume A and B contain strictly positive elements in order to show that A being Morita equivalent to B implies that A and B are stably isomorphic. The argument in their proof [4, Theorem 1.2] depends heavily on results in a paper by Brown that was published earlier that same year. Since then, the statement of the Brown-Green-Rieffel Theorem has restated as below. Many people when referring to the Brown-Green-Rieffel Theorem will refer to this version of their theorem. It is non-trivial to show that a C^*-algebra has a countable approximate unit if and only if it has a strictly positive element.

Theorem 14.66. [12, Theorem 5.55] *Two C^*-algebras A and B which have* **countable** *approximate identities are Morita equivalent if and only if they are stably isomorphic.*

Sketch. First, suppose A and B are stably isomorphic, that is, $B \otimes \mathcal{K} \cong A \otimes \mathcal{K}$. In particular, $B \otimes \mathcal{K}$ is Morita equivalent to $A \otimes \mathcal{K}$. By Exercise 14.64, B is Morita equivalent to $B \otimes \mathcal{K}$ and A is Morita equivalent to $A \otimes \mathcal{K}$. Since Morita equivalence is an equivalence relation, we may conclude B is Morita equivalent to A.

Now, suppose A and B are Morita equivalent. Then there exists a B-A-imprimitivity bimodule E, and thus, $B \cong K(E)$ by Proposition 14.46. As B has a countable approximate unit, E is countably generated as a right Hilbert A-module, i.e., there exists some countable subset $D \subset E$ such that $E = \overline{\text{Span}\{d \cdot a : d \in D, a \in A\}}$ (this is a generalization of the argument that a Hilbert space $\mathcal{H}$ is separable if and only if $K(\mathcal{H})$ has a countable approximate unit). Having E be countably generated is key to leveraging a result about the Hilbert A-module H_A introduced in Example 14.18.

Let $\mathcal{H}$ be a separable Hilbert space, so $\mathcal{K}$ denotes $K(\mathcal{H})$. We want to show that $B \otimes \mathcal{K} \cong A \otimes \mathcal{K}$. By Exercise 14.65, $K(\mathcal{H}) \otimes K(E) \cong K(\mathcal{H} \otimes E)$ and $K(\mathcal{H}) \otimes K(A) \cong K(\mathcal{H} \otimes A)$. By Proposition 14.19, since E and A are both countably generated full right Hilbert A-modules, $\mathcal{H} \otimes E$ and $\mathcal{H} \otimes A$ are isomorphic to H_A, and thus are isomorphic to each other. We get the following:

$$\mathcal{K} \otimes B \cong K \otimes K(E) \cong K(\mathcal{H} \otimes E) \cong K(\mathsf{H}_A) \cong K(H \otimes A) \cong \mathcal{K} \otimes K(A) \cong \mathcal{K} \otimes A.$$

Therefore, B is stably isomorphic to A. □

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