Open Problems from the IPAM Workshop on Symmetric Tensor Categories and Representation theory

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The website for the workshop can be found here. The recordings are available here. We will use the conventions of [EGNO].

1 Problems on STC in characteristic p of moderate growth.

Fix a symmetric tensor category (henceforth STC) $\mathcal{C}$ over an algebraically closed field $\Bbbk$ ([EK], Definition 2.9).

1. (Kevin Coulembier) Let A be a commutative ind-algebra in $\mathcal{C}$. Assume that A is prime, meaning that $0 \subset A$ is a prime ideal (i.e., if $I, J \subset A$ are ideals such that $IJ = 0$ then $I = 0$ or $J = 0$). Is it possible to embed A into a simple, commutative ind-algebra Q_A? This is known to be true for super Tannakian categories. For example, in the category of vector spaces A is prime iff it is a domain, and we can take Q_A to be the fraction field of A.

2. (Kevin Coulembier) Let G be an affine group scheme of finite type in $\mathcal{C}$ (i.e., $G = \mathrm{Spec} B$ where $B = O(G)$ is a finitely generated commutative Hopf algebra in $\mathcal{C}$) and take a closed normal subgroup $H \subset G$ (i.e., a closed subgroup stable under the conjugation action by G). Can we realize H as the kernel of a morphism of affine group schemes $f : G \to K$? Can one define the quotient affine group scheme G/H such that for any affine group scheme K we have a canonical isomorphism $\mathsf{Hom}(G/H, K) \cong \mathsf{Hom}_{H \mapsto 1}(G, K)$, where the latter is the set of homomorphisms that map H to the identity?

3. (Guillermo Sanmarco) Let G be a finite group scheme in the Verlinde category Ver_p, i.e., $O(G)$ is a finite dimensional algebra ([EK], 4.2). Can we decompose G as a semidirect product $G^0 \rtimes \pi_0(G)$ with G^0 infinitesimal ($O(G^0)$ local) and $\pi_0(G)$ étale (i.e., a finite group)?

4. (Guillermo Sanmarco) Develop the theory of Harish-Chandra pairs in STC beyond Ver_p (in the case of Ver_p it is developed by Venkatesh, [Ve]). In particular, do so in the incompressible STC Ver_{p^n} ([BEO]).

5. (Sean Sanford) Develop a theory of root systems for Lie algebras/groups in Ver_p.

6. (Pavel Etingof) Let G be a finite group scheme in Ver_{p^n}. Is the algebra $\mathrm{Ext}^\bullet_G(\mathbf{1},\mathbf{1})$ finitely generated? I expect the answer is yes (this is an instance of my general conjecture with Ostrik that this is true for any finite tensor category, but I have more confidence in this special case and think it is more tractable). This is already interesting for $n = 1$. If G lies in the subcategory of vector spaces, then this is a celebrated result of Friedlander and Suslin [FS], and if it lies in the subcategory of supervector spaces, then it is the generalization of the Friedlander-Suslin theorem due to Drupieski ([Dr]). For $n > 1$ this is non-trivial already for G being the trivial group (i.e., for the Ext algebra of the category itself), but this was proved in [BE], where we actually conjecture a full structure of this algebra (see question 7 in Section 3). Even the case of Ver_4^+ (reduction of superspaces to characteristic 2) for general G would be interesting.

7. (Alex Sherman) Assume that $\mathcal{C}$ is a non-semisimple (symmetric) tensor category that has no non-trivial Serre tensor subcategories with finitely many simples. Can $\mathcal{C}$ have blocks with finitely many simples?

8. (Pavel Etingof) What are polynomial functors in Ver_p, or in Ver_{p^n}, for example for $n = 2$? For vector and supervector spaces they are discussed in [FS],[Dr],[Ax].

   Conjecture. Strict polynomial functors in Ver_{p^n} (and, in fact, in any finite STC) correspond to polynomial representations of the group $\mathrm{GL}(\infty P_1 + \cdots + \infty P_m)$ where P_i's are the projectives/injectives (similarly to [FS]).

   Question. What are the irreducible representations of this group labeled by?

   For Ver_p this problem was solved by Venkatesh ([Ve1]).

   Also we may define the notion of a universal polynomial functor for symmetric tensor categories. This would be a polynomial functor $F_\mathcal{C}$ for every STC $\mathcal{C}$ over a given field $\Bbbk$ (of moderate growth, or arbitrary) which commutes with tensor functors between STC. There are such functors which vanish on the category of (super)vector spaces but not in general categories. For example, if $\mathrm{Fr}_i(X)$ are the components of the Frobenius functor in characteristic p, $1 \leqslant i \leqslant p-1$ ([EO], 1.4), then $\mathrm{Fr}_3(X)$ is such a functor for every $p \geqslant 5$ (it does not vanish on Ver_p). Also in characteristic 2, we have the functor $E(X)$ which is the kernel of the connecting homomorphism $\mathrm{Fr}(S^2X) \to \mathrm{Fr}(\wedge^2 X)$ attached to the short exact sequence

   $$0 \to \wedge^2 X \to X \otimes X \to S^2 X \to 0,$$

   see [EO], Example 3.7; this degree 4 polynomial functor vanishes on vector spaces but not in Ver_4^+. What is the smallest degree of such exotic polynomial functors? (I expect that in characteristic 2 it is 4).

It would be interesting to work out concretely irreducible (universal) polynomial functors of small degree, for example, degrees $p \leqslant d < 2p$, and (perhaps more challengingly) degree $2p$ (the polynomial functors in degrees $< p$ are just the ordinary Schur functors).

2 Problems on STC of non-moderate growth.

Let $\mathcal{C}$ be a STC of non-moderate growth ([EK], 2.6).

1. (Kevin Coulembier) Can there be simple commutative ind-algebra in $\mathcal{C}$ whose module category is not a tensor category? (For example, in vector spaces A is a field, so the category of A-modules is the category of vector spaces over A).

2. (Noah Snyder) Take the category constructed in [KKO] by sending the circle to 1 and the twice-punctured torus to 0. Does this have an abelian envelope?

3. (Nate Harman) Classify simple Lie algebras in the Delannoy category (for the Deligne category $\mathrm{Rep}(S_t)$ this is done in [HK]).

4. (Arun Kannan) What are the irreducible representations of $\mathfrak{gl}(X) = X \otimes X^*$ for X a generator of a non-moderate growth symmetric tensor category? (In $\mathsf{Rep}(\mathrm{GL}_t)$, these are Harish-Chandra bimodules studied in [U]).

5. (Pavel Etingof) Study Davydov-Yetter cohomology of symmetric tensor categories of non-moderate growth (e.g. $\mathsf{Rep}(S_t)$, $\mathsf{Rep}(\mathrm{GL}_t)$, Delannoy, etc.)

   Question. Are there non-braided deformations of $\mathsf{Rep}(S_t)$ other than change of t? In the case GL_t we can deform it to $\mathrm{GL}_{t,q}$, and that's it because $H^3(\mathsf{Rep}\,\mathrm{GL}_t) = \mathbb{C}^2$ (corresponding to varying t and q). N. Snyder has shown that there are no such braided deformations.

3 Problems on support theory.

1. (Peter Webb) Does there exist a non-rigid finite tensor category with infinitely generated cohomology? (Maybe a bad fat point)

2. (Eric Friedlander) How to construct Carlson modules when there is not enough cohomology (e.g. algebraic groups). Infinite-dimensional is okay.

3. (Julia Pevtsova) For (cohomological) support theory for finite groups, finite group schemes, finite dimensional hopf algebras or finite tensor categories, the standard technique to "realize" a closed subset of the spectrum of cohomology is via "Carlson modules', or, equivalently, Koszul objects. Coming up with alternative realization techniques or constructions would be extremely useful for the situations when cohomology is not expected to capture the entire Balmer spectrum of a finite tensor category, such as in the case of Lie superalgebras.

4. (Cris Negron) Can we *do support theory* for finite groups with a tensorial replacement for π-points? To elaborate, given a finite group G and an algebraically closed field k of characteristic p, a π-point for G is a type of flat algebra map to the group ring $\pi : k[t]/(t^p) \to kG$. One then defines the rank variety for a finite G-representation M via the non-projectivity locus

$$\mathcal{V}(M) = \{\pi : k[t]/(t^p) \to kG : M|_\pi \text{ is non-projective}\}/\sim,$$

which lives in the space of all (equivalence classes of) π-points for G. One shows that the space of π-points is identified with the projective spectrum of cohomology $\mathrm{Ext}^*_G(k,k)$, and that the rank variety for M is identified with a cohomological support variety for M, in order to obtain all sorts of results concerning the behaviors of cohomology under tensoring, and a classification of thick tensor ideals in the derived category $D(G)$. This is despite the fact that π-points do not directly acknowledge, or directly use, the tensor structure on $D(G)$!

The question is: Given a finite group G, can we provide an effective mechanism for analyzing cohomology and support in which all operations employed are manifestly tensorial? For example, could we replace π-points with a class of module categories for $D(G)$, and replace the space of π-points with a moduli space of "infinitesimal module categories"?

5. (Cris Negron) Compute $\mathrm{Spec}(\mathrm{perf}\,\mathfrak{X})$ for $\mathfrak{X}$ non-tame DM stack.

Conjecture. Suppose $\mathfrak{X}$ has a good-enough coarse space $\mathfrak{X} \to X$, and let $\mathcal{E}xt_{\mathfrak{X}}(M,N)$ denote the derived inner-Homs for the corresponding action of $\mathrm{QCoh}(X)$ on $\mathrm{QCoh}(\mathfrak{X})$. Then the Balmer spectrum for $\mathrm{perf}\,\mathfrak{X}$ is the relative spectrum of the derived endomorphisms $\mathcal{E}xt_{\mathfrak{X}}(\mathcal{O}_{\mathfrak{X}})$, considered as a sheaf of algebras over X.

6. (Eric Friedlander) Compute Hochschild cohomology of coalgebras.

7. (Julia Pevtsova) Suppose one can compute the Balmer spectrum of the the derived category of a Schur algebra $S(n,d)$, with its internal (Krause) tensor product. Can this be useful in constructing support theories for GL_n-representations, also in some appropriate derived context?

8. (Pavel Etingof + Dave Benson) Compute $\mathrm{Ext}^\bullet(\mathbf{1},\mathbf{1})$ in Ver_{p^n}, and more generally develop support theory. For example, take $p=2$. Consider $\Bbbk[x_1,\ldots,x_{n-1}]$ with $\deg x_i = 1-\frac{1}{2^i}$. Let A be the algebra of elements of integer degree in $\Bbbk[x_1,\ldots,x_{n-1}]$. This is an algebra containing $\Bbbk[x_1^2,x_2^4,\ldots,x_{n-1}^{2^{n-1}}] \subset \mathrm{Ext}^\bullet(\mathbf{1},\mathbf{1})$. For Ver_8, for example, $A = \Bbbk[x,y,z]/(xz-y^2)$, where the degrees of x,y,z are $1,2,3$.

Conjecture. $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^n}}(\mathbf{1},\mathbf{1}) = A$.

Theorem. ([BE]) The reduced scheme $\mathrm{Spec}_{\mathrm{red}}(\mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^n}}(\mathbf{1},\mathbf{1}))$ is isomorphic to $\Bbbk^{n-1}$ with coordinates $y_1 = x_1^2, y_3 = x_2^4, \ldots, y_{2^n-1} = x_{n-1}^{2^{n-1}}$.

4 Problems on modular representations.

1. (Nate Harman)

   Theorem. The following diagram is a co-equalizer diagram in $\mathsf{Tann}_{\Bbbk}$

$$\mathsf{Rep}(GL_n) \underset{\mathrm{Id}}{\overset{\mathrm{Fr}^*}{\rightrightarrows}} \mathsf{Rep}(GL_n) \xrightarrow{q} \mathsf{Rep}(GL_n(\mathbb{F}_p))$$

   Note that the above diagram is *not* a co-equalizer of abelian categories, and this is fundamentally a tensor category statement. This was formulated to make precise and answer a somewhat vague "dream conjecture" Geordie Williamson stated in 2020: That the representation theory of $G(\mathbb{F}_p)$ in defining characteristic is obtained by taking the representation theory of G and 'forcing' the Frobenius to act by the identity.

   Question. Develop a theory of co-limits of STCs and Tannakian categories, and use it to extract homological information about $\mathsf{Rep}(GL_n(\mathbb{F}_p))$ in terms of $\mathsf{Rep}(GL_n)$ and the Frobenius pullback functor. For example, can ext-groups between simples of $\mathsf{Rep}(GL_n(\mathbb{F}_p))$ be computed via a spectral sequence of ext-groups for $\mathsf{Rep}(GL_n)$ reflecting this co-equalizer relationship?

2. (Eric Friedlander) Develop support theory for $G(\mathbb{F}_p)$ in characteristic ℓ where G is a semisimple algebraic group.

3. (Dave Benson, [B]) Let G be a finite 2- group (or a finite 2-group scheme) in characteristic 2, e.g. $G = \mathbb{Z}/2 \times \mathbb{Z}/4$. Let V be an indecomposable representation of odd dimension.

   Conjecture (weak). $V \otimes V^* = \Bbbk \oplus$ even-dimensional indecomposable modules.

   Conjecture (strong). $V \otimes V^* = \Bbbk \oplus$ indecomposable modules of dimension divisible by 4.

5 Miscellaneous problems.

1. (Thorsten Heidersdorf) Develop the theory of tilting modules for quantum supergroups.

2. (Andrew Snowden) Find interesting examples of non-rigid STCs with the Hilbert basis property (don't want examples with fiber functors into rigid categories.) For example, polynomial functors. Want examples with fast growth!

   Hilbert basis property: the symmetric algebra on an object of finite length is Noetherian.

3. (Sophie Kriz) What is the relation between Davydov-Yetter and Quillen cohomology?

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