Problem Session 1 July 15, 2024

1. Solve the following:

(i) If $p \in B(\mathcal{H})$ is a projection, show that $1-p$ is also a projection and $(1-p)\mathcal{H} = (p\mathcal{H})^\perp$.

(ii) For projections $p, q \in B(\mathcal{H})$, show that $pq = 0$ if and only if $p\mathcal{H} \perp q\mathcal{H}$.

2. Let $\mathcal{A}$ be a non-unital Banach algebra. Equip $\mathcal{A}_1 := \mathcal{A} \times \mathbb{C}$ with the following algebra structure: for $x, y \in \mathcal{A}$ and $a, b \in \mathbb{C}$ define

$$(x,a) + (y,b) := (x+y, a+b)$$
$$b(x,a) := (bx, ba)$$
$$(x,a)(y,b) := (xy + bx + ay, ab)$$

Also define

$$\|(x,a)\| := \|x\| + |a|.$$

Show that $\mathcal{A}_1$ is a Banach algebra with the above norm. Does the algebra $\mathcal{A}_1$ possess a unit? If so, what is it? If $\mathcal{A}$ is a C*-algebra, is $\mathcal{A}_1$ again a C*-algebra?

3. Let $x, y \in B(\mathcal{H})$ be self-adjoint. We say $x \geq y$ if $x - y$ is positive semi-definite. This gives us a partial order on the the collection of self-adjoint operators on $B(\mathcal{H})$. Show the following for $x \in B(\mathcal{H})$.

(a) $x, y \geq 0 \Rightarrow x + y \geq 0$.

(b) $x \geq y$, $z = z^* \Rightarrow x + z \geq y + z$.

(c) $x, y \geq 0 \not\Rightarrow xy \geq 0$.

(d) $x = x^* \Rightarrow x^2 \geq 0$.

4. Fill in the following table. (Group exercise.) Note that some cells can have more than one correct answer.

	Algebraic Definition	Spatial Definition
Normal	$TT^* = T^*T$	
Self-Adjoint	$T = T^*$	
Projection	$T = T^2 = T^*$	T is an orthogonal projection onto a closed subspace of H
Invertible		
Unitary	$T^*T = I = TT^*$	
Isometry		$\|T\xi\| = \|\xi\|$ for all $\xi \in \mathcal{H}$
Co-Isometry	$TT^* = I$	
Partial Isometry	$T = TT^*T$	For some closed subspace $\mathcal{K} \subset \mathcal{H}$, $T\|_{\mathcal{K}}$ is an isometry and $T\|_{\mathcal{K}^\perp} \equiv 0$

5. By Corollary 8.11 of the prerequisite notes, $FR(\mathcal{H})$ is a two-sided $*$-ideal in $B(\mathcal{H})$. Show that it is not a norm-closed ideal.

6. Let $S \in B(\ell^2(\mathbb{N}))$ be the shift operator

$$S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$$
$$(a_1, a_2, \ldots) \mapsto (0, a_1, a_2, \ldots)$$

(a) What is S^*?

(b) What are the source and range projections?

(c) Show S is an isometry, $S^*S = 1$, but not a unitary ($S^*S = 1 = SS^*$).

7. Let $\{p_n : n \in \mathbb{N}\} \subset B(\mathcal{H})$ be a family of pairwise orthogonal projections.

(a) For $m < n$ and $\alpha_m, \alpha_{m+1}, \ldots, \alpha_n \in \mathbb{C}$, show that

$$\left\| \sum_{j=m}^{n} \alpha_j p_j \right\| = \max_{m \le j \le n} |\alpha_j|.$$

(b) For $(\alpha_n)_{n\in\mathbb{N}} \in c_0(\mathbb{N})$, show that

$$\left(\sum_{j=1}^{n} \alpha_j p_j \right)_{n\in\mathbb{N}}$$

is a Cauchy sequence (with respect to the metric induced by the operator norm).
(What does this allow us to define?)