# Problem Session 2 July 16, 2024

**C*.1** Prove that C*-norms are unique. That is, if $\|\cdot\|$ and $\|\cdot\|_1$ are two norms with respect to which $A$ is a C*-algebra, then $\|a\| = \|a\|_1$ for all $a \in A$.

**C*.2** The follow problems are to help you build intuition for $C_0(X)$ where $X$ is locally compact and Hausdorff (or $C(X)$ where $X$ is compact and Hausdorff).

(a) What is the $C^*$-algebra structure on $C_0(X)$ (i.e., what is the norm, addition, multiplication, and involution)? Show $C_0(X)$ is a $C^*$-algebra. When is $C_0(X)$ unital? What is the unit?

(b) What is the spectrum of an element $f \in C(X)$?

(c) What are the projections in $C_0(X)$?

**C*.3** Prove the following proposition: Suppose $\{a_1, ..., a_n\} \subset A$ are linearly independent and $\{b_1, ..., b_n\} \subset B$. Then

$$\sum_1^n a_i \odot b_i = 0 \implies b_i = 0, \text{ for all } 1 \leq i \leq n.$$

If $\{e_i\}_{i\in I}$ is a basis for $A$ and $\{e'_j\}_{j\in J}$ is a basis for $B$, then $\{e_i \odot e'_j\}_{(i,j)\in I\times J}$ is a basis for $A \odot B$.

**C*.4** For any (not necessarily commutative) C*-algebra $A$, the Gelfand transform $\Gamma : A \to C(\hat{A})$ is an algebra homomorphism. Moreover, $\|\Gamma(a)\| \leq \|a\|$.

**W*.1** Show that if a net $(x_i)_{i\in I} \subset B(\mathcal{H})$ converges in operator norm to some $x \in B(\mathcal{H})$, then it converges in the strong operator topology to $x$. Show that if a net $(x_i)_{i\in I} \subset B(\mathcal{H})$ converges in the strong operator topology to some $x \in B(\mathcal{H})$, then it converges in the weak operator topology to $x$.

**W*.2** Consider the shift operator $S$ on $\ell^2(\mathbb{N})$:

$$S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots).$$

Show that $((S^*)^n)_{n\in\mathbb{N}}$ converges to zero in the SOT, but $(S^n)_{n\in\mathbb{N}}$ does not.

**W*.3** In this exercise you will show that taking adjoints of normal operators is continuous with respect to the strong operator topology.

(a) Show that $y \in B(\mathcal{H})$ is normal if and only if $\|y\xi\| = \|y^*\xi\|$ for all $\xi \in \mathcal{H}$.

(b) Suppose $(x_i)_{i\in I} \subset B(\mathcal{H})$ is a net of normal operators converging to a normal operator $x \in B(\mathcal{H})$ in the strong operator topology. Show that $(x_i^*)_{i\in I}$ converges to $x^*$ in the strong operator topology. [Note: you may want to show that taking adjoints is continuous with respect to the weak operator topology.]

**Pre-Req.0** (Warning! This problem is a bit long.)
Show that $FR(\mathcal{H}) \subset L^1(B(\mathcal{H}))$, $\mathrm{Tr}(\theta_{\xi,\eta}) = \langle \xi, \eta \rangle$ and $\|\theta_{\xi,\eta}\|_1 = \|\xi\|\|\eta\|$ for $\xi, \eta \in \mathcal{H}$. [Hint: You should look at the prerequisite notes to get see how every element in $FR(\mathcal{H})$ looks like.]

**Related Problems:**

1. Show that $(x_i)_{i\in I} \subset B(\mathcal{H})$ converges to $x \in B(\mathcal{H})$ in the strong operator topology if and only if $((x - x_i)^*(x - x_i))_{i\in I}$ converges to zero in the weak operator topology.

2. Consider the operator $S_n \in B(\ell^2(\mathbb{N}))$ defined by

$$S_n e_j := \begin{cases} e_{j+1} & \text{if } 1 \leq j \leq n-1 \\ e_1 & \text{if } j = n \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Determine the formula for $S_n^*$ and show that $S_n$ is normal.
   (b) Show that $(S_n)_n$ converges in the strong operator topology to the shift operator $S$, but $(S_n^*)_n$ does not converge to $S^*$.
   (c) Reconcile the previous part with Problem W$^*$.3.

3. Describe the projections in $C_0(X)$ when $X$ is
   (a) $(0, 1]$
   (b) $[0, 1]$
   (c) $[0, 1/3] \cup [2/3, 1]$
   (d) $[0, 1/3) \cup (2/3, 1]$

4. (a) Show that all ideals in $C(X)$ are of the form

$$\{f \in C(X) : f|_F = 0\}$$

   for some closed subset $F$ of $X$.
   (b) What are the maximal ideals in $C(X)$?

5. If $A_1, A_2$ are $(^*)$-algebras and $\psi_i : A_i \to B$ $(i = 1, 2)$ are $(*)$-homomorphisms, when is $\psi_1 \otimes \psi_2$ a $(*)$-homomorphism?