White Paper: “Geometry, Statistical Mechanics, and Integrability” (IPAM Long Program, Spring 2024)

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Executive Summary

This document summarizes the activities and outcomes of the Long Program "Geometry, Statistical Mechanics, and Integrability," held at the Institute of Pure and Applied Mathematics (IPAM) from March 11 to June 14, 2024. We also briefly explore current open questions and future directions in the field.

In the last 20-30 years, probability theory and statistical mechanics have been revitalized with the introduction of various tools, notably conformal geometry and discrete analyticity, as well as algebraic geometry and integrable systems.

Many familiar statistical mechanics models, such as the Ising model, dimer model, spanning tree model, and their cousins, have an intrinsic underlying geometric structure. For example, discrete analytic geometry was used by Kenyon, Lawler, Schramm, Werner, Smirnov, and Chelkak in their proofs of conformal invariance of scaling limits. Other work on the dimer model has led to connections with hyperbolic geometry, Lorentzian geometry, and symplectic geometry.

Recent connections between classical and discrete geometric structures on surfaces and combinatorial models such as the dimer and six-vertex models have revealed a significant connection to integrable systems and discrete geometry.

There are well-known connections between statistical mechanics models, algebraic combinatorics and representation theory: they have a common interest in Young tableaux, Gelfand–Tsetlin patterns, Knutson–Tao puzzles, and Littlewood–Richardson coefficients and their generalizations, for example. The Bethe Ansatz and Yang–Baxter equations were developed for the six-vertex model but are now fundamental tools in combinatorial representation theory, also giving explicit connections with integrability.

The application of conformal field theory (CFT) to statistical mechanics has been another pivotal area of research. CFT has proven to be a powerful tool for describing the scaling limits of critical lattice models. Schramm Loewner Evolution (SLE) and its variants were instrumental in understanding conformal invariance and scaling limits. Extending these methods to higher-rank models remains a complex and open question.

The program brought together many researchers from this somewhat disparate realm of ideas, united by the underlying themes of geometry and statistical mechanics. Activities included introductory tutorials, four workshops, and eight working groups. The groups studied recent literature and open problems in areas such as adapted geometric embeddings, vertex models, and asymptotic algebraic combinatorics.

Multiple directions for future research emerged from the program. Higher rank versions of statistical mechanics models have mysterious and unexplored connections with more sophisticated CFTs including W-algebras, as well as deeper connections with representation theory. Recent progress connecting the six-vertex model with symmetric polynomials gives a new direction of exploration towards a better understanding of asymptotic behavior of structure constants and other algebraic quantities.

The progress connecting geometries to free fermionic models leads us to analogous questions for the six-vertex model and beyond: is there an appropriate discrete geometry underlying any critical statistical mechanics model? There is clearly much to be explored!

1. The Geometry of Statistical Mechanics Models

1.1. Introduction

Statistical mechanics is a branch of mathematical physics whose goal is to derive macroscopic properties of matter starting from the microscopic behavior of elementary constituents. Local interactions may produce long-range order. Notable models include the Ising model for magnetism and percolation describing how a fluid traverses a porous medium. At a microscopic level, systems are discrete and possess random behavior. At the macroscopic level, remarkable structures can emerge, exhibiting many symmetries (invariance by scaling, rotation, and more generally conformal transformations).

A breakthrough in understanding these symmetries was the development of conformal field theory in physics in the ‘70s and ‘80s. However, this early theory lacked mathematical rigor in describing how to pass from the discrete setting to the continuum. A major challenge for mathematicians has been to bridge this gap, starting with the case of two dimensions where tools of complex analysis are available in the continuum. One useful tool in the discrete setting has been to define an appropriate discrete geometry and a discretized version of complex analysis.

1.2. Embeddings

The height function of the dimer model is expected to converge to the Gaussian Free Field with a certain conformal structure predicted by Kenyon and Okounkov. A new idea is to obtain this conformal structure by embedding the dimer graph in the “correct” way before taking limits, similarly to how random walk on a graph converges to Brownian motion when the graph is embedded using Tutte’s harmonic embedding.

One of the reading groups was dedicated to the study of recent works by Chelkak, Kenyon, Lam, Laslier, Ramassamy, and Russkikh on new embeddings called t-embeddings of planar graphs carrying the dimer model. These t-embeddings simultaneously generalize isoradial embeddings, Chelkak’s s-embeddings for the Ising model and Tutte’s harmonic embeddings. They reveal the correct conformal structure of the dimer model and allow to effectively generalize the notion of discrete holomorphicity developed by Chelkak, Hongler, Izyurov, Kenyon, Smirnov and others. The theory of t-embeddings has already brought many new results on scaling limits of the dimer model.

The group also discussed non-probabilistic aspects of t-embeddings, including the connections to circle patterns and Miquel dynamics, T-systems, and to discrete isothermic surfaces in Lorentz space.

1.3. Discrete geometry

In recent years, efforts from the three communities of statistical mechanics, discrete integrable systems, and discrete differential geometry have converged to produce a common framework integrating their fields. This approach, pioneered by Affolter, George, Glick, Izosimov, Pylyavskyy, and Ramassamy, is based on graphs whose vertices encode geometric objects (points, lines, etc.) and whose edges encode incidence relations between these objects. Notable examples include t-embeddings and s-embeddings from statistical mechanics, the pentagram map from discrete integrable systems and Q-nets from discrete differential geometry. Dynamics on such objects rely on incidence theorems in classical geometry and the Goncharov-Kenyon dimer integrable system. A highlight of *Workshop I* was the presentation of a similar framework by Fomin and Pylyavskyy, giving a general mechanism to derive linear incidence theorems, recovering known ones and producing a host of new ones.

1.4. Recurrence relations

A recurrence relation is a rule that produces a set of numbers given some initial data (such as the Fibonacci sequence). The octahedron recurrence (also known as discrete Kadomtsev–Petviashvili (dKP) equation) and cube recurrence (also known as discrete BKP) are classical recurrences dating to the work of Dodgson and Kennelly in the 19th century. Together with the hexahedron recurrence (also known as discrete CKP) defined by Kashaev, they form a rich family of recurrences that appear in many parts of mathematics.

Solutions of these recurrences have a remarkable property called the Laurent Phenomenon - they are Laurent polynomials in the initial data rather than general rational functions, an observation that was a precursor to the introduction of cluster algebras by Fomin and Zelevinsky. In the last two decades, these solutions were shown by Carroll, Kenyon, Melotti, Pemantle, and Speyer to have a natural combinatorial interpretation in terms of dimers and related objects like spanning trees and Ising configurations. Petersen, Speyer and others have exploited this connection to compute limit shapes for these models.

It is an interesting question to extend these combinatorial formulas. This involves several ingredients: First there is the twist map, which is a geometric automorphism of the Grassmannian. Second, there are webs and multiple dimers which give new combinatorial models. Finally, there is the immanant construction, which is a natural involution on web spaces. Combining these tools we can perform many concrete computations.

1.5. Spectral transform

A reading group focused on the spectral transform originally introduced by Kenyon and Okounkov, and recent progress relating the dimer model to the geometry of higher genus Riemann surfaces. The spectral curve of a dimer model is the vanishing locus of the determinant of the bi-periodic (magnetic) Kasteleyn operator, whose inverse determines the correlations of the model and may be expressed via discrete analytic techniques. In the periodic case, Kenyon, Okounkov and Sheffield showed that it classifies the phases of the dimer model.

Initial discussions centered around the genus-zero case of the dimer model considered by Kenyon on infinite, planar, isoradial graphs, i.e. those for which each face is inscribed in a circle of equal radius. These graphs have a special role in physics and probability, e.g. in Liouville gravity isoradial graphs can be interpreted as a discretization of flat metrics in the plane.

The interest of the group was to understand the extension of this result to higher genus curves due to Fock. His approach constructs a dimer model on minimal planar graphs, generalizing isoradial graphs, using theta functions on higher genus spectral curves. In the periodic case, all dimer models are gauge equivalent to a model arising from the above construction. A striking feature of this classification, interesting from the point of view of cluster-dynamics, is that the spider move is equivalent to Fay's trisecant identity from the theory of theta functions.

The latest developments on the subject include the probabilistic implications of Fock's construction considered by Boutillier, Cimasoni and de Tilière and the study of limit shapes by Boutillier, Cimasoni and de Tilière, Berggren and Borodin, Bobenko, Bobenko and Suris, and the extension to the spanning tree model by George.

1.6. Outlook

The problems mentioned above were discussed in *Workshops I, II* and *IV*. Some questions remain for future research:

1) The existence of t-embedding for arbitrary finite bipartite graphs.
2) What is the solution for the general octahedron recurrence for arbitrary bipartite graphs?
3) Construction of a theta-function inverse for the spectral transform of the Ising model. What is the theta-function identity that realizes the Ising Y-Delta move?

1.7. Research groups at IPAM

The problems mentioned in this sectio were discussed in the research groups "Embeddings", "Elliptic Dimers" and "Higher Octahedron Recurrence".

The members of the "Embeddings" group were: N. Affolter, T. Alcalde López, T. Berggren, V. Boskovic, C. Boutillier, P. Di Francesco, T. George, R. Kedem, M. Lis, C. Meng, M. Nicoletti, M. Russkikh, J. Scott, M. Tikhonov, HT. Vu.

The members of the “Elliptic Dimers” group were: T. Alcalde López, M. Basok, T. Berggren, C. Boutillier, V. Boskovic, T. George, C. Meng, M. Nicoletti, M. Russkikh, J. Scott, M. Tienni, M. Tikhonov.

The members of “Higher Octahedron Recurrence” group were: N. Affolter, V. Boskovic, A. Chavez, L. Foster, T. George, I. Le, C. Meng, H. Mularczyk, M. Russkikh, J. Scott, M. Tienni.

2. Limit Behavior of Statistical Mechanical Models

2.1. Introduction

The word “universality” describes the phenomenon by which a discrete system, with potentially complicated local rules, converges in the large-size limit to a much simpler, *universal* continuous system. The paradigmatic example of universality is the random walk (on a periodic graph in the plane or in space), which converges in the large-time limit to Brownian motion, a continuous random process that can be described without reference to any underlying discrete structure.

Universality is proven or conjectured to be present in virtually all *critical* statistical mechanics systems: when tuned to the critical temperature or some other critical parameter, systems defined on lattices have continuous scaling limits which are independent of the underlying lattice on which they are defined, often exhibiting rotational, scaling, and even conformal symmetries.

Universality is very hard to prove mathematically, but intense work over the last 25 years has led to a number of beautiful examples: for example, in the dimer model, the 2d spanning tree model, and the 2d Ising model. Within the dimer model, we know that (in some generality) the height function converges to the Gaussian Free Field; the boundary behavior along a polygonal edge converges to the GUE Corners Process; along the arctic boundary the fluctuations converge to the Airy Process (as discussed by L. Zhang during the program). In the spanning tree model, tree branches converge to the Schramm-Loewner Evolution $SLE(2)$. In the critical Ising model, spin domain boundaries converge to $SLE(3)$, and the spin field has a particular scaling limit field, a so-called “minimal model” of conformal field theory.

2.2. Limit shapes

Discrete models in statistical mechanics often concentrate, after scaling, to deterministic shapes. As a basic example, consider a biased random walk $x(t)$ on the real line which at each time t moves to the right with probability p and to the left with probability $1 - p$ independently of the past moves. This process converges to *linear motion* in the scaling limit with velocity $2p - 1$.

In two dimensions, limit shape examples are: minor processes of unitarily invariant random matrix ensembles (with connections to free probability), height functions from random standard Young tableaux, and height functions of dimer models. These limit shapes satisfy a variational principle (they minimize a certain convex "energy'', typically arising from entropic considerations) and thus satisfy a PDE. In *integrable* cases, this PDE can be solved using analytic or, more interestingly, piecewise analytic functions.

A recently analyzed model that relates to both the Horn problem in linear algebra and Littlewood-Richardson coefficients from algebraic combinatorics is that of Knutson-Tao hives. Integer-valued hives are height functions for square triangle tilings, which degenerate, respectively, to Gelfand-Tsetlin patterns and to real-valued hives in two different limits.

While limit shapes beyond 2d seem very hard, recent works on limit shapes for 3d systems were presented by R. Kenyon at *Workshop I* and C. Wolfram at *Workshop III*.

2.3. Determinantal processes

Among the most accessible random spatial processes are the Gaussian ones: computing their observables reduces to linear algebra. In practice, however, most important models are not Gaussian, and we need other tools to analyze them. An important class of processes, in some sense the next simplest after the Gaussian ones, are determinantal processes.

A process is *determinantal* if its multipoint correlations are given by subdeterminants of a certain operator. The existence of such an operator allows one to fully determine the multipoint observables of the model. Of particular interest to this program has been the determinantal nature of planar dimer models and non-intersecting path processes. For example, T. Berggren and A. Borodin presented methods for utilizing determinantal structure in doubly periodic Aztec diamonds in *Workshops I* and *IV*.

Important to understanding the behavior of a determinantal model are the correlation functions, which can be easily computed if a suitable form of the determinantal operator is known. Correlation functions describe how spatially separated points in the model affect each other. For example, in the domino tiling model, the further away dominoes are from one another, the less correlated they are. There are two caveats to this statement. Firstly, dominoes can be largely affected by their proximity to the boundary. Secondly, correlations may decay at different speeds depending on the macroscopic region in which the dominoes are located.

Knowledge of the correlation function allows us to understand the macroscopic behavior of the limit shapes discussed above. We are able to classify different phases of the model and compute exact algebraic curves for the phase boundaries. Beyond that, we are also able to understand the local fluctuations along these boundaries. The behavior along these boundaries can be shown to converge to Airy processes and Pearcey processes.

2.4. Yang-Baxter integrability

In the hierarchy of mathematical models of random surfaces, determinantal processes are included in a wider class of *Yang-Baxter integrable* models. These systems can model complex behavior, such as traffic or random growth processes.

This emerging class is characterized by quantum integrability arising from the Yang-Baxter equation (YBE; also known as the Y-Delta move). The YBE is a class of local equivalence transformations preserving the partition function (probability normalizing constant). YBE provides access to global quantitative information (such as asymptotic free energy) about the system under large space- or time-scaling. The YBE was among the original tools employed by Onsager, Lieb, Yang, Baxter, and others since the mid-20th century in studying Ising and ice type (six-vertex) models and quantum spin chains.

The YBE provides just enough symmetry to express quantities of interest exactly, thereby accessing their asymptotics. It also leads to interesting couplings and differential equations for particle systems, as presented by L. Petrov in *Workshop IV*. YBE integrable models have led to significant progress in our understanding of the universality of random growth and stochastic interacting particle systems.

However, YBE solvability is typically restricted to one-point observables, that is, expectations of functions depending on a single particle. A new research direction (see below) combines this exact approach with analytic tools, such as strong characterization of limiting objects and the method of Gibbsian line ensembles, to obtain multipoint limit theorems for certain stochastic interacting particle systems. In the future, we hope that these methods can be applied to non-exactly solvable systems, leading to a more complete picture of universality.

2.5. KPZ

The *Kardar-Parisi-Zhang (KPZ) universality class* originated from describing various natural growth processes, such as the growth of bacterial colonies and molecular condensation. KPZ statistics have been discovered in many other probability models. For example, the *Tracy-Widom distribution*, which governs the one-point fluctuation of the KPZ growth process, is also the law of the largest eigenvalues of many random matrix models, including the adjacency matrix of random graphs (which models social networks) and covariance matrices in multivariate statistical analysis.

Many of the statistical mechanics models discussed in this program have been proven to exhibit KPZ universality behaviors under certain scaling limits. These include various planar dimers (domino or lozenge), six-vertex models, and various stochastic interacting particle systems, whose current fluctuation can be encoded as growing surfaces. As alluded to, proofs of these KPZ behaviors mostly originated from obtaining exact solutions using tools such as determinantal structures and YBE. A recent example is on the colored 1d asymmetric exclusion process, presented by I. Corwin in *Workshop IV*.

On the other hand, in many particular problems, the method of deriving exact solutions is not perfectly suitable for establishing universality, as the exact structures are usually analyzable only in special settings and are sensitive to perturbations. There are still major unresolved questions that require new ideas and approaches. Concrete examples include last passage percolation with general weights and certain stochastic interacting particle systems with site-dependent or time-dependent disorder. Another example is in dimers, where the arctic boundary behavior is expected to be Airy, which is in the KPZ universality class, while the exact solutions can only be analyzed for specific boundary conditions.

A strategy for extending limit results from special to general settings is to use analytic methods to compare them, showing that their differences are of a smaller order. This approach is inspired by works that establish universality in random matrix theory (by

Erdos, Yau, etc.) and has been successfully used to obtain KPZ and other types of universality (GUE corners, Pearcey), in some generality, in certain dimer models. For various statistical mechanics models discussed in this program, many limit results have been obtained using exact solutions in the past decades (and still ongoing). These limit behaviors include but are not limited to KPZ. It would be interesting to understand their universality by combining formulas and analytic approaches.

2.6. Outlook

The problems mentioned above were discussed in all of the four workshops during the program.

Some of the big questions remaining in this area are: How universal is KPZ? What other universality classes occur in statistical mechanics models? To what extent can limit shape theory be extended to higher dimensional models? Is there an appropriate discrete geometry underlying any critical statistical mechanics model, beyond dimers and six-vertex models?

3. Algebraic Combinatorics and Integrability

3.1. Introduction

Algebraic combinatorics studies objects and quantities motivated by algebra, representation theory, and algebraic geometry. Such objects include Young tableaux and symmetric functions (notably the Schur functions). The main problems concern formulas for algebraic quantities like structure constants, dimensions, etc. Initially, the field was propelled by beautiful exact formulas (e.g., the hook length for the number of standard Young tableaux, which also gives the dimension of irreducible S_n modules), and powerful but intricate bijections (e.g., the Robinson-Schensted-Knuth correspondence (RSK)). At this point, however, we have seemingly exhausted the realm of such exact formulas but are still far from understanding all the important structure constants. Hence, we need to understand their behavior asymptotically, a problem that could employ methods and ideas from statistical mechanics. On the other hand, algebraic combinatorics tools were helpful in many aspects of mathematical physics, from understanding lozenge tilings to more general vertex models. Our goal has been to expand this interaction. These topics and other connections were the main subjects of *Workshop II.*

3.2. Structure constant asymptotics

Structure constants describe the expansions of a product of elements of distinguished bases. The basis of Schur polynomials has rich connections to the geometry of the Grassmannian as well as the representation theory of S_n. Their structure constants are the Littlewood–Richardson (LR) coefficients, which count particular semistandard Young tableaux as well as Knutson–Tao honeycombs, Berenstein–Zelevinsky triangles, and certain Gelfand–Tsetlin patterns. Inspired by the LR coefficients, Murnaghan (1938) introduced Kronecker coefficients, multiplicities of the decomposition of tensor products of irreducible representations of S_n into irreducible components. Unlike the LR case, there is still no positive combinatorial rule for computing them, and their computation remains nontrivial.

The asymptotics of Kronecker and LR coefficients have been long studied, with their maxima given by Stanley (1995). For LR coefficients, the shapes attaining these maxima were found by Pak, Panova, Yeliussizov (2018). They similarly establish the existence of shapes attaining these maxima for the Kronecker case. In general, we need more precise upper and lower bounds for the LR case when partitions grow in certain regimes. Upper bounds for the LR case were established in some cases (see Narayanan, Sheffield (2024), Belinschi, Guionnet, Huang (2022)), but the answers are inexplicit. For Kroneckers, finding an explicit family for which the asymptotics hold remains open.

These problems have important connections with *free probability* as studied by Biane (1998). For instance, one approach to the asymptotics of LR coefficients is to analyze free convolution of their underlying limit shapes. As noted by Pak, Panova, Yeliussizov (2018), this is related to the description of the shapes which give the equality case for some natural inequalities holding among hook integrals. We plan to further investigate these connections towards characterization of partitions attaining the corresponding maxima.

The basis of Schubert polynomials generalizes the Schur setting. However, Schubert structure constants have no combinatorial rule. Unlike Schurs, there are far fewer identities for Schuberts and their structure constants. Thus, the asymptotics of these Schubert coefficients are poorly understood. In future work, we hope to determine new identities for Schuberts and leverage them to understand Schubert asymptotics.

Another direction is the study of dimensions of irreducible representations, which are equivalent to enumeration of standard Young tableaux. Following work of Vershik and Kerov (1985) on typical dimensions of irreducible representations of S_n, Kerov and Pass

(1992) conjectured, that if a partition of n maximizes the dimension for a given n, the logarithm of this quantity is asymptotically $\sqrt{n!}e^{-c\sqrt{n}}$. The problem is to show that such c exists. While this remains open, Yeliussizov and Pak were able to improve known numerical bounds for c. Further, the precise asymptotics of skew SYT of shape λ/μ when the two partitions approach limit curves is still not fully understood, the second order terms are no explicit formulas in full generality. Using variational principle techniques Istvan Prause obtained the more precise asymptotics $\sqrt{n!}e^{-cn}$ with explicit constant c when the shapes approach the Plancherel shape.

3.3. Stochastic particle systems

Integrability provides powerful tools to study stochastic particle systems, where specific features of the system, for example, the distribution of particles on the integer lattice, can be encoded in explicit formulas. This approach has been instrumental in discovering new connections between probability, representation theory, and symmetric functions via the works of Borodin, Corwin, Gorin, Ferrari, Petrov, Sasamoto, and others. The study of interacting particle systems leads to universal asymptotic phenomena.

Among different stochastic particle systems, the q-deformed totally asymmetric simple exclusion process (q-TASEP) is particularly interesting and is closely connected to Macdonald operators. Specifically, the action of the q-Whittaker operators on q-Whittaker symmetric functions allows one to explore the distribution of a particle's position via its q-moments. Studying symmetries of this family of observables leads to couplings with stochastic particle systems involving time-translated q-Whittaker operators acting in different sets of variables. Finding rank recursion relations (powering the couplings) may be useful in finding new integrability in other models, including random matrices.

3.4. Macdonald operators

Macdonald polynomials are a family of symmetric polynomials which are uniquely defined as common eigenfunctions of the Macdonald operators. Macdonald polynomials reduce to Hall-Littlewood polynomials, q-Whittaker functions, or Schur polynomials under certain specializations of the parameters. In these reductions, the corresponding Cauchy identity can be proven in an "RSK-like" manner. It remains open to find a "bijective" (RSK-style) proof of the Cauchy identities in the general Macdonald case. Macdonald operators were instrumental in computing expectation values of certain observables of Macdonald processes, and a better algebraic understanding of their action is desirable.

The Macdonald operators are q-difference operators, which, together with operators of multiplication by functions, generate the spherical Double Affine Hecke Algebra (DAHA). So far, all DAHA constructions and most of the known results are for a fixed number of variables (rank), which appears naturally in the algebraic setting. However, it is interesting to consider Macdonald operators in different ranks and investigate relations between them. Focusing first on q-Whittaker operators, members of the group proved that the time-translated q-Whittaker operators of different ranks satisfy recursion relations. Similar relations for the time-translated Macdonald operators follow as well. Such relations should help to understand the relations between q-Whittaker or Macdonald processes with "swapped" particles.

3.5. Vertex models

Vertex models, major players of *Workshop IV*, are two-dimensional lattice models whose configurations are obtained by equipping edges of a subgraph of the lattice with arrows (states) subject to local constraints around common vertices. Local vertex weights can be chosen to be *integrable* (satisfying the Yang-Baxter Equation), which leads to exact solutions.

Apart from a rich theory involving representations of various algebras, vertex models can be used to tackle concrete combinatorial and probabilistic problems such as 1) the enumeration of alternating sign matrices (ASMs), 2) obtaining exact expressions for LR coefficients as partition functions with rational integrable weights, 3) deriving expressions of non-symmetric Macdonald polynomials, and 4) exactly computing expectations of observables in stochastic particle systems, in a form amenable to asymptotic analysis.

Let us focus on combinatorial applications of vertex models. See above for probabilistic applications. A well-known open problem in algebraic combinatorics concerns the maximal value for the principal specialization of Schubert polynomials. Stanley (2017) shows that the maximal value is between $2^{n^2/4}$ and $2^{n^2/2}$. A year later, Morales, Pak, Panova narrowed this to between $2^{0.29n^2}$ and $2^{0.37n^2}$. The upper bound follows from the number of ASMs. Generalizing Schuberts, Grothendieck polynomials are partition functions over all ASMs with nonlocal weights depending on a parameter β. The same maximal specialization problem holds for general β. Setting $\beta = 1$, the model simplifies, giving exact specialization $2^{n(n-1)/2}$.

Dimer models can typically be identified with systems of non-intersecting paths, while vertex models allow the paths to interact. The difference is best seen in their asymptotic behavior. With specific boundary conditions, scaling limits display arctic curve phenomena. In dimers, arctic curves are analytic (in great generality), while they are piecewise analytic for vertex models. The exact nature of this distinction has yet to be understood. Studies of correlations, such as emptiness formation probabilities, should help tackle fluctuations around arctic curves in vertex models. At the enumerative level, surprising connections exist between vertex models (six-vertex, twenty-vertex) and rhombus/domino tilings on suitable domains, which extend to portions of their limit shapes. A theoretical framework behind these connections is still lacking and could potentially follow from further investigation of height functions and their surface tension.

3.6. Lozenge Tilings

Lozenge tilings are tilings of regions in the plane using the three types of rhombi on the triangular grid (equivalent to a dimer model on the hexagonal grid). There is a celebrated bijection between lozenge tilings of hexagons and 3d stacks of cubes pushed into a corner of a room, obtained by simply viewing the lozenges as a perspective drawing. We can describe these stacks using plane partitions, 2d generalizations of the (line) partitions that are central objects in algebraic combinatorics. These 3d stacks of cubes are coded by 2d contour maps. Using the Lindström–Gessel–Viennot lemma, we can enumerate them via a determinant, giving MacMahon's formula. The same lemma proves the equivalence between the combinatorial and determinantal definitions of Schur functions.

We can generalize the problem by replacing the hexagon by another domain. One can ask asymptotic questions about these models such as classifying typical random lozenge tilings in different domains. Partition functions of lozenge tilings of special domains coincide with particular symmetric functions. Some open problems on lozenge tiling enumeration can be found in Lai (2021).

3.7. Matroids and polyhedral geometry

Independently introduced by Nakasawa and Whitney (1930's), matroids are combinatorial objects that generalize the notion of linear independence. From a matroid one can encode the maximally independent sets as vertices of a convex polytope. Questions as fundamental as computing volume or enumerating faces of various dimensions are complex and combinatorially rich. For example, the lattice point enumerating function, *L(t),* counts the number of integer points in dilates of a polytope. A fundamental result of

Ehrhart (1962) proves *L(t)* is a polynomial, so we call *L(t)* the Ehrhart polynomial. If the coefficients are positive, the polytope is called Ehrhart positive and the coefficients have potential to encode meaningful combinatorial data. Understanding the Ehrhart theory of matroid polytopes, when they are Ehrhart positive, and what combinatorial interpretations exist (both polytopal and matroidal) are directions of interest. Similar questions arise in Weighted Ehrhart theory, introduced during *Workshop II*, which attaches a weight function to a polytope.

3.8. Bijections of dimer configurations

Many combinatorial objects turn out to be equinumerous. One way to prove this is by using formulas that enumerate both objects and check that they match up. A preferable way (when possible) is to construct an explicit and natural bijection between the objects, especially when equinumerosity is already known. However, there is no universal approach for finding bijections. During the non-workshop week seminar, a talk by S.H. Byun presented a bijection between dimer configurations of two related graphs. This led to a discussion about finding families of graphs for which the number of dimer configurations of one family is less or equal to another and showing this by constructing an injective map from the smaller family to the larger one.

3.9. Clone Schur measures on Fibonacci words

Parallel to the representation theory and algebraic combinatorics of symmetric groups (mediated by the Young lattice of partitions), there exists a variant based on Stanley's Young-Fibonacci (YF) lattice. Like the Young lattice, the YF lattice is an example of a differential poset and, therefore, supports a version of the Robinson-Schensted correspondence. Furthermore, the Robinson-Schensted correspondence is reflected in the structure theory of a family of algebras introduced by Okada, whose Bratelli diagram is the YF-lattice. There are also analogs of the Schur functions, called clone Schur functions. The IPAM program has seen progress in probabilistic applications of the YF lattice, worked out by Scott and Petrov. They classified positive specializations of clone Schur functions, which entails analyzing a strong version of total positivity for semi-infinite tridiagonal matrices. This concept of positivity can be studied vis-à-vis the Stieltjes moment problem (characterizing probability measures by their moments). Accordingly, some positive specializations relate to well-known discrete orthogonal polynomials (such as Charlier) and their q-deformations. Positive specializations lead to interesting probability measures on binary words, whose scaling limits are related to Residual Allocation (Griffiths-Engen-McCloskey) type distributions on the infinite-dimensional simplex of summable real-valued sequences. These distributions previously appeared in mathematical models of population genetics and machine learning.

3.10. Research groups at IPAM

The problems mentioned above were discussed in the research groups “Asymptotic Algebraic Combinatorics” and “Macdonald operators” and the reading group “Schur processes.”

The members of “Asymptotic Algebraic Combinatorics” were: Q. Francois, I. Le, H. Mularczyk, H. Narayanan, G. Panova, C. Robichaux, H. Walsh, D. Yeliussizov, C. Zhao. Topics discussed include Schubert calculus, free probability approaches to studying LR coefficients, convex geometry and variational principles, saturation problems about structure constants, formulas for the number of skew standard Young tableaux.

The members of “Macdonald processes” were: J. de Gier, P. Di Francesco, A. Gunna, W. Guo, R. Kedem, W. Mead, G. Panova, L. Petrov, J. Scott, HT. Vu, D. Yeliussizov, Z. Zhou. Topics discussed include stochastic particle system, Q-system, Macdonald Operators, q-TASEP with different initial data, clone symmetric functions, contour integral ansatz for q-TASEP, vertex models.

The members of “Schur processes” were: F. Colomo, P. Di Francesco, T. George, W. Guo, I. Le, W. Mead, J. Scott, S. Tata, Z. Zhou. The group concentrated on reading the paper of A. Okounkov and N. Reshetikhin on Correlation function of Schur process (2001).

4. Conformal Field Theory and Higher Rank Models

4.1. Introduction

In the last few decades, mathematicians have realized the fundamental importance of proving conformal invariance for critical lattice models in the scaling limit. This led to the breakthrough works of Schramm, Werner (Fields Medal, 2006), Smirnov (Fields Medal, 2010), and recently Duminil-Copin (Fields Medal, 2022). These works related complex analysis and probability theory more closely to questions on conformal invariance in physics. However, these results only include specific models, where remarkable symmetries and local relations in the lattice level are available.

The study of critical phenomena in statistical mechanics models has been an active area of research for a long time in the physics community. Ever since its inception in the 1980s, *conformal field theory* (CFT) has proven to be a powerful tool to describe the scaling limits of such statistical mechanics models. The realm of CFT has a wide reach in mathematics, and there is much recent progress in clarifying and making more rigorous its precise relation to statistical mechanics.

In CFT, correlation functions are governed by an algebraic structure, the Virasoro algebra. In many cases, including cases in higher rank (in contrast to rank 1 where there is only the Virasoro symmetry), one can find more general structures, including W-algebras, Temperley–Lieb algebras and other planar algebras, and logarithmic phenomena in CFT. It is desirable to study these structures in concrete lattice models and see how we can identify, either directly on the lattice or in the scaling limit, how these structures show up and how they could be used to gain insight into the related statistical physics models. Further work will allow us to apply powerful techniques from CFT to persistent questions in stochastic models and thereby to applied sciences as well.

4.2. CFT and SLE

One of the major breakthroughs in the mathematics of conformal invariance was the introduction of Stochastic Loewner Evolution by Schramm (1999), now known as Schramm–Loewner Evolution (SLE), which led to mathematical derivations of critical exponents and rigorous descriptions of scaling limits of critical interfaces. It became evident that SLE is a universal object closely related with 2D CFT. A loop version of SLE, termed the Conformal Loop Ensemble (CLE), was introduced by Sheffield and Werner, and helped describe scaling limits of loops in various models of interest. Using these random processes, the interfaces in double-dimers and various other models have been understood to a great extent. However, in the higher rank cases, the geometric interfaces have a much more complicated structure: they involve branching (and are thus termed webs) and it is not clear at all how to describe them in terms of generalizations of SLE and CLE processes.

We can get a deeper understanding of conformal invariance for critical lattice models by identifying limit objects with meaningful quantities in CFT. This will provide insight into the conceptual reasons of universality, beyond ad-hoc lattice considerations. SLE analysis has turned out to be helpful here, e.g. boundary-to-boundary crossing probabilities can be described using interface-connectivity properties, even when the lattice model is not exactly solvable.

4.3. CFT and Statistical Mechanics

Arguably, the most studied lattice model is the *Ising model*, introduced in the 1920s as a 2D model of ferromagnetism. Soon after the inception of CFT in the 1980s, it was (conjecturally) observed in the physics literature that the scaling limit of the Ising model can be described by a conformal field theory. Yet even in this ubiquitous well-studied model, only a fraction of the CFT content has been established in the scaling limit.

One of the famous successes of CFT in the treatment of statistical models is the Cardy–Smirnov formula for percolation crossing probabilities; written down by Cardy using CFT techniques in 1992 and rigorously proven by Smirnov in 2001. Interestingly, the CFT that is needed to derive such a formula belongs to the class of CFTs known as *logarithmic CFTs*, whose name stems from the fact that their correlation functions possess logarithmic singularities.

In CFT, the complete reducibility of representations of the chiral algebra of interest means that the theory is *rational*. Otherwise, and this is by far the most common case, the theory is *logarithmic*. It has been observed in some cases that such theories are suitable to answer probabilistic questions concerning non-local observables in lattice models, including polymers and percolation.

4.4. Higher Rank Models

On a bipartite graph, an n-dimer cover is a superposition of n-dimer covers. An n-dimer cover can also be interpreted as an $SL(n)$ web subgraph of the original graph.

If one imposes boundary behavior on the dimers, one can ask for the partition function of n-dimers with given boundary conditions. For double dimers, we recover the connection probabilities found by Kenyon and Wilson. For n-dimers, it is natural to use reduced webs to parameterize the different possible connection events. An important question is to calculate connection probabilities in finite graphs as well as their scaling limits.

We should mention two other connections here. Firstly, the multinomial tiling model is a generalization of multidimers studied by Kenyon and collaborators. Secondly, webs are used to study Laurent expansions of cluster variables on Grassmannians. This is related to the octahedron recurrence discussed elsewhere.

Another Perspective. The dimer model scaling limit is related to the theory of free fermions. From the perspective of coupling to gauge fields, the scaling limit of the dimer model is the 'Dirac Fermion' CFT. Both the multi-dimer model and the multi-component Dirac Fermion can be coupled to gauge fields. A $U(1)$ gauge field can be used to measure correlations of the height function field of a single-dimer model. Kenyon and Dubédat used an $SL(2)$ gauge field to study double-dimers. $SL(n)$ gauge fields and n-fold dimers give relationships between topological types of web configurations on surfaces. The continuum limit analogs can be identified with the analogous twisted partition functions of the Dirac Fermion.

For n-dimers, there is also an associated height function, generalizing the height function for double-dimers. These can be studied in the gauge field framework by choosing a special choice of 'abelian' connections. These height functions might limit to a W-algebra analogue of the Gaussian Free Field. This would be natural from the point-of-view of the Dirac fermion, which gives a well-known construction of the W-algebra.

Loop and web models. When using q-deformed versions of webs, one can define so-called loop $O(N)$ models and web models. The loop $O(N)$ model is a paradigmatic integrable model and has been much studied in the last decades. Its configurations are given by non-intersecting loops on the hexagonal lattice. The Boltzmann weight consists of a local part but also of a non-local one that assigns a weight N to each loop. Recently, there has been progress in understanding the logCFT governing an important set of correlation functions, in particular geometrical ones, as well as connection probabilities in these models. Progress has been made by studying various fusion channels in 4-point functions, making explicit use of the global $O(N)$ symmetry.

On the other hand, loop models can be generalized in a natural way. The transfer matrices of the loop model can be phrased in terms of morphisms in the Temperley–Lieb category. This category is equivalent to the category of representations of $U_q(sl_2)$. Analogous categories of webs describe representations of higher rank quantum groups. One can define (Yang–Baxter) integrable models whose configurations are given by webs embedded in the hexagonal lattice. The fluctuations of their associated height functions can be described non-rigorously by an imaginary Toda CFT. This leads to natural conjectures about the scaling limit of geometric events such as connection probabilities in terms of correlation functions in CFT.

4.5. Outlook

The topics mentioned above were discussed in *Workshops I* and *IV*.

It will be very interesting to study the scaling limits of n-dimers. Since SLE curves and the Virasoro algebra are related, in the higher rank case one expects to obtain generalizations of SLE curves related to W-algebras. We also expect to reveal further connections with real enumerative geometry, in particular with the famous Shapiro–Shapiro conjecture. A couple of years ago, Peltola and Wang gave a novel proof of the conjecture for the Grassmannian $Gr(2, n)$ by studying $SLE(\kappa)$ curves connecting boundary points as κ approaches 0 (semiclassical limit). The general case of the Shapiro–Shapiro conjecture should be related to $SLE(0)$ curves associated to W-algebras and semiclassical limits of web models.

4.6. Research groups at IPAM

Some of the topics described in this section were discussed in the research group *Conformal Field Theory and Statistical Mechanics.* The participants of this group were, alphabetically, D. Adame-Carrillo, N. Aghaei, T. Alcalde-López, Y. Feng, E. Peltola, A. Lafay, W. Ruszel, S. Tata and H. Wu.

Topics discussed include the symplectic fermions CFT, discrete Gaussian Free Field, and the relationship to the scaling limit of the dimer model and uniform spanning tree models.

5. Work in progress at IPAM

We list manuscripts in progress on which the program participants have worked at IPAM. The program's participants are in **bold.**

- O. Abuzaid, V. Healey, and **E. Peltola**. *“Large deviations of Dyson Brownian motion on the circle and multiradial SLE(0+).”*
- O. Abuzaid and **E. Peltola**. “*Large deviations of radial SLE(0+).”*
- **D. Adame-Carrillo,** *“Symplectic fermions in bounded domains.”*
- **D. Adame-Carrillo, W. Ruszel,** *“Discrete symplectic fermions and its scaling limit CFT.”*
- F. Ardila-Mantilla, **A. Chavez**, J. De Silva, P. E. Harris, J. L. Herrera Bravo, A. R. Vindas-Meléndez. *”Valuation polytopes of height two posets.”*
- **N. Berestycki**, **M. Lis**, M. Liu, and **E. Peltola**. *“Uniform spanning trees and random matrix statistics.”*

- **T. Berggren, M. Nicoletti, M. Russkikh.** *"Perfect t-embeddings of the uniformly weighted hexagon."*
- **C. Boutillier, B. de Tilière,** *"Fock's dimer model on the Aztec diamond."*
- F. Camia and **Y. Feng**, *"Logarithmic correlation functions in 2D critical percolation",* arXiv: 2403.18576.
- **F. Colomo,** A.G. Pronko. *"Evaluation of integrals for the emptiness formation probability in the square-ice model,"* arXiv:2405.04358.
- **P. Di Francesco, L. Petrov, HT. Vu.** "*q-Whittaker and Macdonald time-translated operator rank recursion.*"
- **P. Di Francesco, HT. Vu.** *"T-system with slanted initial data and t-embedding"*
- J. Dousse, **A. Morales, G. Panova, L. Petrov, D. Yeliussizov.** *"Random Pipe Dreams via Integrable Probability."*
- **Y. Feng**, M. Liu, **E. Peltola, H. Wu,** *"Multiple SLEs for k\in (0,8): Coulomb gas integrals and pure partition functions",* arXiv: 2406.06522.
- **Y. Feng**, **E. Peltola** and **H. Wu**. *"Crossing probabilities of multiple percolation interfaces: Generalizations of Cardy's formula and Watts' formula."*
- A. Karrila, **A. Lafay**, **E. Peltola**, and J. Roussillon. *"UST branches, fused SLE(2), and $c = -2$ degenerate conformal blocks."*
- **R. Kenyon, I. Prause**, *"Random antichains in the cube."*
- A. Knizel, **L. Petrov.** *"Random Lozenge Waterfall: Dimensional Collapse of Gibbs Measures."*
- **A. Lafay**, **E. Peltola**, and J. Roussillon. *"Fused Specht polynomials and $c = 1$ degenerate conformal blocks."*
- **A. Lafay, I. Le.** *"Connection probabilities for n-dimers in the scaling limit."*
- **G. Panova, L. Petrov.** *"Hook-length Formulas for Skew Shapes via Contour Integrals and Vertex Models."*
- **E. Peltola** and A. Schreuder. *"Loewner traces driven by Lévy processes."*
- **L. Petrov, J. Scott.** "*Clone Schur measures on Fibonacci words and their scaling limits.*"
- **S. Ramassamy**, B. Terlat. *"Wall-crossing for a hydrodynamic limit of the infinite bin model."*
- **M. Tikhonov.** *"From Yang-Baxter equation to Busemann functions for random polymers."*